A DYNAMIC MODEL OF LABOR SUPPLY

UNDER UNCERTAINTY

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Research Paper No. 272

July 1985

Previous versions of this paper were presented at the 1983 summer Econometric Society meetings, in the author's thesis, and in an appendix to Flinn and Heckman (1982). This paper developed out of joint work with James Heckman. The author's dissertation research was supported by the Social Science Research Council dissertation grant. Points of view or opinions stated herein do not necessarily represent the official position of the Social Science Research Council.

INTRODUCTION

This paper develops an optimizing model of individual labor supply that admits reasonable technological and informational constraints. It concentrates on the all-or-nothing employment decision. Coleman (1984, chapter 3, and 1985b) presents evidence that most of the cyclical fluctuations in aggregate hours worked are the result of movements in and out of employment rather than marginal adjustments in the hours of work. Given this evidence, it is reasonable to concentrate on the employment decision. Focusing on individuals' participation decisions, however, means that one cannot use a representative agent to model the aggregate behavior; individuals' behavior must be correctly aggregated. This model addresses aggregation. A result of this model is that the technological and informational constraints (probabilities of finding job openings) enter the worker's decision problem. This model provides a framework for analyzing both the behavioral response and the welfare effects of changes in these constraints. In a general equilibrium setting, these constraints may act as "prices" to equilibrate the labor market.

This model views individual behavior in the labor market as a process of moving between labor force activities. This distinguishes it from the standard approach, where the outcome (employment or unemployment) and not the process (movement between employment and unemployment) is modeled. Given the substantial monthly gross flows between labor force states (see, e.g. Marston, 1976; Abowd and Zellner, 1985), this is a natural approach to the labor market. Agents are in a stochastic environment and make decisions whether they want to work or not. When an agent decides he wants to work, however, he must search for a job. Jobs at the going wage cannot be found immediately, and an agent must spend time and (possibly) money looking for a firm with an available job. The probability of finding an available job in the next instant, if less than unity, acts as a constraint on labor supply; a worker would work at the current wage but is unable to do so because jobs are not instantaneously available. This is a formal model of frictional unemployment, although one could also label such unemployment involuntary. (For simplicity the model focuses on employment, and the hours decision is ignored. When working all agents work at the same intensity. For present purposes this is acceptable.)

This model is closely related to the analysis of the duration of unemployment. (See, e.g. Kaitz, 1970; Salant, 1977; Nickell, 1979; Heckman and Singer, 1984a, 1984b.) In those papers the focus is more on the empirical distribution of leaving times for a single spell of unemployment, rather than a theoretical model of the decision process. The relationship between the empirical distribution and models of the decision process is, however, quite close. The regularities uncovered in the empirical studies provide both a guide and a check on theoretical work. In addition, the econometric techniques used often transfer to the estimation of parameters of the theoretical models. As Heckman and Singer (1984a) have pointed out, the functional forms used for hazard functions in empirical studies sometimes have little theoretical foundation. In the best of all possible worlds, economic models of the decision process provide functional forms for the hazards and often restrictions across hazards for different states. (As Heckman and Singer have also pointed out, model derived hazards often have the disadvantage of being overly restrictive, or not being in closed form and thus computationally burdensome.)

The model presented in this paper gives hazards as functions of the parameters of the worker's optimization problem. The estimation of these parameters proves to be quite easy. Research in the area of dynamic discrete choice models is active. Other models that deal with employment versus non-employment decisions are Lippman and McCall (1976); Toikka (1976); Burdett and Mortensen (1978a, 1978b). The present model is a form of the search models discussed in Lippman and McCall, and is directly related to the three state model of Toikka. Rust (1984) provides a good discussion of estimation of a general class of controlled discrete choice models (to which, however, this model does not belong). Some general characteristics tie all the dynamic discrete choice models together.

- They model the choice process, with the outcomes being determined probabilistically.

- Changes occur between discrete states (say an individual going from employment to unemployment).

- The models are dynamic, with decisions occurring through time. Although not true of all models mentioned above, in this model changes can occur at any point in time, so there is no natural time unit (weeks, say, or months).

- Various economic factors affect the process. It is the purpose of theory to specify how these affect the process, and thereby the realized outcomes.

When specified in continuous time, these are examples of general discrete-state, continuous-time, stochastic models. Models of this sort have been used widely in the physical sciences to study, e.g., radioactive decay and chemical reactions. The general formulation is

- there are s states indexed by i, $i = 1,2, ..., s$.

- the probability of occupying state i at time t is $p_i(t)$.

- there is a transition rate at time t between pairs of states. This is called $q_{ij}(t)$, and is defined as the limit, as Δ approaches zero, of the ratio of the transition probability to Δ:

$$q_{ij}(t) = \lim_{\Delta \to 0} P[i \to j \text{ in time } \Delta | \text{ in } i \text{ at } t] / \Delta$$

- the probabilities of occupying states, $p_i(t)$, are governed by a system of differential equations:

$$dp_i(t) / dt = \Sigma_i^s \, p_j(t) q_{ji}(t)$$

with $q_{ii}(t) = -\Sigma_j \, q_{ij}(t)$.

In matrix notation, this is

$$\underline{p}'(t) = \underline{p}(t)Q(t)$$

where $\underline{p}(t)$ is the row vector of state probabilities at time t. The economic models mentioned above model all or some of the elements of the transition matrix Q. This model does the same. The elements of Q are hypothesized to be the result of agents solving a continuous time optimization problem in a stochastic environment.[1]

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[1] The model in this paper is actually a continuous-state, continuous-time, stochastic model. In general the mathematics necessary to analyze such models is more difficult than that for discrete-state models. In particular, the system of differential equations for $p_i(t)$ now becomes an integro-differential equation on the function $p(x,t)$, where x is the state variable. I largely ignore this problem because simplifications of the model, used in specific applications, reduce the problem to a discrete-state problem.

DEVELOPMENT OF A DYNAMIC MODEL OF LABOR SUPPLY

In this model workers face a known wage, but must wait for an offer to arrive. A worker knows the distribution of what is available, but does not know which of several firms currently has an opening. A worker who wants a job visits an employment office periodically (say once a week, or at random times that average out to once a week) to find out if a job is available. When a job is available the worker takes it. If a job is not available, he must return later. (Staying at and visiting the employment office costs, so workers visit at discrete times.) People have some control over their search intensity (how often they visit the employment office), by choosing to be unemployed or out of the labor force, but the probability of finding a job opening is exogenous. (Firms have control over how many job openings there are by posting job notices at the employment office more or less frequently.) In this model both the wage and the probability of finding a job enters the worker's value function. The standard labor supply model assumes that only the wage matters: Any worker who wishes to work instantaneously takes a job, and the wage is the only variable entering into the labor supply and labor demand decisions. In contrast, in this model the labor supply decision depends on both the offered wage and the probability of being hired. Workers face technological constraints on their labor supply.

The model is based on individual behavior, and so any aggregate behavior must be derived by correctly aggregating from the micro to the macro level. Each individual solves the same optimization problem, but at any point in time individuals have different opportunities; i.e. different draws of market and non-market benefits. This endogenously generates heterogeneity (both observed and unobserved), which means that a representative agent approach to aggregation will not work. In addition this model focuses on the all-or-nothing decisions of employment and unemployment. The aggregation problems must be faced. Although this makes the mathematics of the model more difficult, it provides a richer and more fruitful model.

The worker can be in one of three labor force activities: working, unemployed (searching), or not in the labor force (NLF). A worker receives instantaneous utility of consumption in each activity. In addition, the stochastic environment generates new opportunities, which then lead to future expected benefits (resulting from either continuing in the same activity or changing to another). The utility and future opportunities are:

1. employment
    - Instantaneous benefit is utility of consuming wage: u(w).
    - Future opportunities are 1) layoff into unemployment or 2) arrival of high non-market benefits and quit into NLF.

2. Unemployment
    - Instantaneous benefit is the disutility (cost) of searching: u(-c).
    - Future opportunities are 1) new job offer and 2) new non-market benefits high enough to induce dropping out of the labor force.

3. NLF
   - Instantaneous benefit is the utility of consuming non-market benefits: u(x).
   - Future opportunities are 1) new job offer which may induce transition to employment (depending on current value of non-market benefits and 2) new non-market benefits which may be low enough to induce transit to unemployment.

New wage and non-market offers are drawn from known distributions. Layoffs and new offers arrive at Poisson-distributed random times. These distributions and arrival rates may change, however, according to the state of the economy.

The mathematical state of the system occupied by the worker is given by the values of the wage (w), the non-market benefits (x), and the state of the economy (z). New (w,x,z) states are assumed to arrive at Poisson distributed times. In other words the probability of an arrival in a small period Δt is equal to $\Delta t + o(\Delta t)$. These arrival probabilities are:

1. NLF benefits arrive at rate μ independent of the worker's activity. When an offer X arrives, it is drawn from the (known) distribution G(x), which may vary with the state of the economy, z.

2. Job offers arrive at rate λ_3 in the search activity, λ_2 in the NLF activity, and λ_1 in employment. λ_i may vary with the state of the economy, z. When a new wage offer W arrives, it is drawn from the (known) distribution F(w) (which again may depend on z, and in addition may depend on the current wage, w).

3. Layoffs arrive at the rate δ when a worker is employed. Layoff is into the unemployment state. Layoff rates may depend on z, the state of the economy.

4. New states of the economy arrive at rate ν. When a new state Z(z), arrives, it is drawn from the distribution $F_z(z)$, which may depend on the current value of z, but not on any worker's (w,x).

The worker has utility of consumption u(c). For simplicity the worker is not allowed to borrow or lend, and so must consume everything immediately. When a worker is laid off, he receives unemployment benefits until he either transits to NLF or finds a job. The current rewards, therefore, are

1. u(w) when employed with wage w
2. u(x) when NLF with benefits x
3. u(k-c) when laid off and receiving unemployment benefits
4. u(-c) when unemployed (having entered from NLF)

The value of being employed, NLF, and unemployed can be written as $V_1(w)$ $V_2(x)$, $V_3(k)$. The optimal value can be written as

$$V(w,x,k,z) = \text{Max}[V_1(w,z),V_2(x,z),V_3(k,z)].$$

The task now is to formulate expressions for V_1, V_2, V_3, and then prove that the problem is well defined and has a solution. To start with employment, the current value is $V_1(w,z)$. Bellman's equation states that today's value must equal the discounted expected value from an instant in the future. In an instant Δt the probabilities of arrival are

1. $\mu\Delta t+o(\Delta t)$ that a new NLF offer X arrives (expected value $EV(w,X,0,z)$)
2. $\delta\Delta t+o(\Delta t)$ that a layoff occurs (value $V(0,0,k,z)$)
3. $\nu\Delta t + o(\Delta t)$ that a new state of the economy, Z, arrives (value $V(w,0,0,Z)$)
4. $\lambda_1\Delta t + o(\Delta t)$ that a new wage offer w' arrives (value $V(\max(w,w'),0,0,z)$)
5. $1-\mu\Delta t-\delta\Delta t-\lambda_1\Delta t-\nu\Delta t+o(\Delta t)$ that nothing happens (value $V_1(w,z)$)

Writing this down gives

$$V_1(w,z) = (1+r\Delta t)^{-1}[u(w)\Delta t + \mu\Delta tEV(w,X,0,z) + \delta\Delta tV(0,0,k,z) + \nu\Delta tEV(w,0,0,Z) + \lambda_1\Delta tEV(\max(w,W),0,0,z) + [1-(\mu+\delta+\nu+\lambda_1)\Delta t]V_1(w,z)] + o(\Delta t)$$

Rearranging and letting $\Delta t\rightarrow 0$ gives

$$V_1(w,z) = (\mu+\delta+\nu+\lambda_1+r)^{-1}[u(w) + \mu EV(w,X,0,z) + \delta V(0,0,k,z) + \lambda_1 EV(\max(w,W),0,0,z) + \nu EV(w,0,0,Z)] \tag{1}$$

Similar arguments for V_2 and V_3 gives

$$V_2(x,z) = (\mu+\lambda_2+\nu+r)^{-1}[u(x) + \mu Ev(0,X,0,z) + \lambda_2 EV(W,x,0,z) + \nu EV(0,x,0,Z)] \tag{2}$$

$$V_3(k,z) = (\mu+\lambda_3+\nu+r)^{-1}[u(k-c) + \mu EV(0,X,k,z) + \lambda_3 EV(W,0,k,z) + \nu EV(0,0,k,Z)] \tag{3}$$

Finally, the optimized value, $V(w,x,k,z)$ is

$$V(w,x,k,z) = \text{Max}[V_1(w,z),V_2(x,z),V_3(k,z)] \tag{4}$$

Under relatively mild conditions, $V(w,x,k,z)$ is well defined, continuous, and increasing in (w,x) for each z. In particular, $V(w,x,k,z)$ exists and is well behaved for $u(w)=w$, i.e. for wealth maximization. See appendix A for a proof.

What is important for the transition process between activities is the probability that a particular offer will be either accepted or rejected. If the distributions $F(w)$ and $G(x)$ do not depend on the current (w,x) value (so that offers are uncorrelated) the model generates a reservation wage function for each z. When employed at wage w, a worker will exit to NLF whenever he receives an offer above a critical value $x^*(w)$. This is the reservation value $x^*(w)$. The function $x^*(w)$ is determined by setting the value of a job equal to the value of NLF:

$$\text{(5a)} \quad V_1(w,z) = V_2(x,z)$$

For a given w (and z) this gives a unique x, $x^*(w,z)$, since $V_1(w,z)$ is not a function of x, and $V_2(x,z)$ is increasing in x.

The lowest wage observed will be that wage which makes a person indifferent between working and searching further; i.e., w^*_0 is the solution to

$$\text{(5b)} \quad V_1(w,z) = V_3(0,z) \; .$$

Any wage above w^*_0 will induce a searcher without unemployment benefits to take a job. Similarly, any wage above w^*_k, the solution to

$$\text{(5c)} \quad V_1(w,z) = V_3(k,z)$$

will cause a searcher with unemployment benefits to take a job. The lowest acceptable NLF offer, n^*_0, solves

$$\text{(6a)} \quad V_2(x,z) = V_3(0,z)$$

and n^*_k solves

$$\text{(6b)} \quad V_2(x,z) = V_3(k,z) \; .$$

When wage offers are correlated, there may not be a reservation wage function. Refer back to (5a). For any x, there will be a set of w values, $\{w: V_2(x,z)<V_1(w,z),\ x \text{ and } z \text{ given}\}$, which cause transition from NLF to employment. The set may not be connected. For example, a low wage offer may signal nothing about future offers, and so will be accepted. Similarly, a high offer may signal nothing about future offers, and so also will be accepted. Some intermediate offers, however, may indicate that there is a high probability of a yet higher offer, and so will not be accepted in anticipation of these higher offers. Essentially, $V_2(x,z)$ is then a function of the current wage offer, in that the current wage offer, while it cannot be accepted once rejected, helps predict future wage offers. Thus (5a) becomes $V_1(w,z)=V_2(x,z,w)$, so that both V_1 and V_2 are functions of the current wage. The argument that V_1 is increasing in w and V_2 does not depend on w no longer holds. For the rest of the paper, except where noted, it will be assumed that wage offers are not correlated. In addition, unemployment benefits will be assumed zero, so that the reservation value x^*_k disappears.

This model is related to other models of discrete choice over time, such as Toikka (1976) and the class of models discussed in Rust (1984). There are important distinctions, however. First, this model is stated in continuous time, Toikka and Rust are in discrete time. Second, and more importantly, new wage and values of non-market time arrive independently, and when a non-market offer arrives in the job state, the job is still retained as an option. In Toikka's model, both wage and non-market offers arrive simultaneously, and the worker must choose between the better of the two. Toikka's formulation has the result that the future payoff is a constant independent of the current wage or non-wage value, while in the present model the future payoff depends on the current wage and non-wage values. In a model where wage and non-wage offers arrived simultaneously, the value of a job would be something like:

$$V_1(w) = (\gamma_1+r)^{-1}[u(w) + \gamma_1 EV(W,X)]$$

The term $\gamma_1 EV(W,X)/(\gamma_1+r)$ represents the future payoffs, and does not depend on the current value of w (although it does depend, through γ_1, on being in the job activity). The constant $EV(W,X)$ appears in the expression for the value of each activity. This considerably simplifies the expressions for reservation wages. It allows one to calculate comparative statics results, but at the price of the unrealistic assumption that offers arrive simultaneously and that the future payoff is independent of the current wage. Assuming simultaneous arrivals also assures that model transition rates depend only on observables, such as the current wage or the current activity -- employment, search, or NLF. Rust (1984) imposes assumptions which assure that transition rates do not depend on unobserved state variables. In the present model with independent arrivals, both the current wage and unobserved non-market values become state variables. In other words, unobserved heterogeneity is generated. Transitions are Markovian conditional on these state variables, but transitions are not Markovian conditional on the wage or current activity (employment, unemployment, or NLF) alone. The model with independent arrivals can generate declining hazards if one conditions on the current activity alone. For example, the hazard for exit from unemployment will be declining if those with unemployment benefits have a lower exit rate and so remain unemployed longer; the classic mover-stayer problem.

The solution of the worker's optimization problem gives the value of being in any particular state, and a set of decision rules on when to transit from one state to another. The decision rules are summarized by a set of reservation wages:

$x^*(w,z)$ = reservation value for transition from job to NLF This is the minimum value of non-market benefits which induces worker being paid w to quit and drop out of the labor force (When wage offers are correlated there will be a possibly disconnected acceptance set $\{x:V_1(w,z)\geq V_2(x,z) w$ and z given$\}$.) This can also be written as $w^*(x,z)$, and then represents the minimum acceptable wage when in NLF with value of non-market time x.

x^*_0 = reservation value for transition from unemployment to NLF. This is the minimum value of non-market benefits which induces an unemployed worker to drop out of the labor force. It will in general depend on z.

w^*_0 = reservation value for transition from unemployment to job. This is the minimum acceptable wage offer. It will depend on z.

Behavior is governed by the following transition rates:

eu: job to unemployment = layoff rate (exogenous)
en: job to NLF = (rate of arrival of NLF offers) * P[non-market offer is above the reservation value $x^*(w,z)$]
ue: unemployment to job = (rate of arrival of job offers in the unemployment state)*P[$W \geq w^*_0$]
un: unemployment to NLF = (rate of arrival of non-market offers) * P[a given non-market offer is above the reservation value x^*_0]
ne: NLF to employment = (rate of arrival of job offer in the NLF state) * P[the current non-market benefits are below $x^*(w)$]
nu: NL to unemployment = (rate of arrival of non-market offers) * P[a given non-market offer is below the reservation value x^*_0]

Transitions between activities (employment, unemployment, and NLF) are the basic behavior that this model describes. The transition rates are the product of exogenous arrival rates for new offers, and endogenously determined decision rules which determine when the offer is acceptable (is above the reservation value). The observed behavior is determined by the interaction of the chosen reservation wages and the exogenous arrival rates.

The optimization problem enters the transition process through the following probabilities involving the reservation wage functions (or acceptance sets in the case where reservation wages do not exist):

$$(7) \qquad P[X<x^*_0] = G(x^*_0) = q_0$$

$$P[X>x^*(w)] = 1 - G(x^*(w)) = q_2(w)$$

For expositional simplicity, I will take the case of z fixed and a single wage with uncorrelated offers; the distribution $F(.)$ has mass p at w and $1-p$ at 0. The basic results for identification and estimation will hold for any discrete distribution of wages. (The case of a continuous wage distribution will be left for future research. Although the value function and the optimal policy exists, as stated previously, estimation is more difficult. The process is then a continuous-state, continuous-time stochastic process.)

The solution of the worker's optimization problem generates transition probabilities. The observed behavior, however, consists of actual transitions between labor force activities. The task, then, is to go from the transition probabilities to the probabilistic behavior of the individual, and from there to the aggregate behavior. In this simplified version of the model unemployment behavior is the same no matter how one entered or how long one has been there. For NLF, however, behavior depends on the (unobserved) current values of x. For transitions between NLF and unemployment what matters is where a new X offer is relative to x^*_0. For $x<x^*_0$, no one stays NLF, and

workers prefer to be unemployed. When employed, what matters is where a new X offer is relative to the reservation value $x^*(w)$. For $X>x^*(w)$, the worker quits into NLF, while for $X\leq x^*(w)$, the worker rejects the offer and stays with the job. Similarly, someone NLF accepts a job as long as his current value of non-market benefits is below $x^*(w)$.

The probability of transition from one activity into another is the product of the probability that the exogenous state (the x value) changes, times the probability that the new state will be such as to make a transition optimal. For example, in a small period of time Δt, the probability of a new non-market offer is $\mu\Delta t+o(\Delta t)$. When currently employed at wage w, the probability that the offer is high enough to cause transition from employment to NLF is $P[X>x^*(w)]$. The probability of a transition from employment to NLF in time Δt is $\mu P[X>x^*(w)]\Delta t+o(\Delta t)$. Similarly, the probability of transition from e to u is $\delta\Delta t+o(\Delta t)$. From this one can infer that the exit rates from employment (wage w) are:

eu = rate (employment to unemployment) = δ
en = rate (employment to any NLF state) = $\mu P[X>x^*(w)] = \mu q_2$
en(x) = rate (employment to NLF with x) = $\mu g(x)I[x>x^*(w)]$

Similarly the exit rates from unemployment and NLF are

$ue = \lambda_3 p$
$un = \mu P[X>x^*_0] = \mu(1-q_0)$
$un(x) = \mu g(x)I[x>x^*_0]$
$ne(x) = \lambda_2 pI[x<x^*(w)]$
$nu = \mu P[X\leq x^*_0] = \mu q_0$
$nn(x_1,x_2) = \mu g(x_2)I[x_2>x^*_0]$

Behaviorally, the continuum of NLF states collapses into two states, n_1 and n_2. The n_2 state includes those who are truly out of the labor force, and will not take a job if offered. The n_1 state, however, includes people who are not searching intensively for a job (and so do not report themselves unemployed) but would take a job if offered. This corresponds to both the common feeling that the distinction between unemployment and NLF is artificial, and to the Bureau of Labor Statistics (BLS) definition of unemployment. This model predicts that some of those NLF (the n_1-types) will behave like those unemployed, while others (the n_2-types) are distinctly different from those unemployed. Essentially, the difference between n_1-types and those unemployed is a difference of degree -- the intensity of job search. The BLS defines unemployment as those available for work and who "had made specific efforts to find employment during the prior 4 weeks." (BLS 1980, p.1) Thus, n_1 types are just those who are searching at a low intensity.

The transition rates generate a set of simultaneous linear differential equations which describe the probabilistic behavior of an individual. The differential equations (still for a single wage and fixed z) are:

$$e' = -(\delta+\mu q_2)e + \lambda_3 pu + \lambda_2 pn_1$$

$$u' = \delta e - [\lambda_3 p + \mu(1-q_0)]u + \mu q_0 n_1 + \mu q_0 n_2$$

$$n_1' = \mu(1-q_0-q_2)u - [\lambda_2 p + \mu(q_0+q_2)]n_1 + \mu(1-q_0-q_2)n_2$$

$$n_2' = \mu q_2 e + \mu q_2 u + \mu q_2 n_1 - \mu(1-q_2)n_2$$

The probabilities of being in each activity can be written as

(8) $$p(t) = [e_t, u_t, n_{1t}, n_{2t}]$$

and then the differential equations can be written compactly as

(9) $$p' = pQ \ .$$

The transition matrix is Q:

(10)

	e	u	n_1	n_2
e	$-(\delta+\mu q_2)$	δ	0	μq_2
u	$\lambda_3 p$	$-(\lambda_3 p+\mu(1-q0))$	$\mu(1-q0-q_2)$	μq_2
n_1	$\lambda_2 p$	$\mu q0$	$-(\lambda_2 p+\mu(q0+q_2))$	μq_2
n_2	0	$\mu q0$	$\mu(1-q0-q_2)$	$-\mu(1-q_2)$

These are the parameters that enter in the differential equations generating observed transitions, and are the parameters of the model which must be estimated.

When the wage distribution consists of two wages, (w_1,w_2), the transition matrix is 6x6. The wage is now a state variable, so that employment actually consists of two distinct groups, those with w_1 and those with w_2. The continuum of NLF states collapses into three states: n_1-types who will accept any job if offered, n_2-types who will accept a w_2 offer but reject a w_1 offer, and n_3-types who will reject any job offer. For example, the differential equation governing the behavior of a person currently employed at wage w_2 is

$$e'_2 = \lambda_1 f(w_2)e_1 - [\delta+\mu P[X>x^*(w_2)]]e_2 + \lambda_3 f(w_2)u + \lambda_2 f(w_2)(n_1 + n_2) \ .$$

Naturally, this assumes that the lower wage (w_1) is such that individuals with zero value of non-market time (those unemployed) will choose to work at that wage. If this were not the case, the model would be equivalent to that discussed above, with wages of 0 and w.

For a discrete wage distribution with finite or countably infinite points of support, there will be some lowest acceptable wage, w_1. Corresponding to each wage w_i at or above w_1, there will be a reservation value of non-market time, $x^*(w_i)$. Any individual with $x^*(w_i-1)<x\leq x^*(w_i)$ will accept a wage offer of w_i or above, but reject offers below w_i. This will define n_i types. In other words, there will be as many reservation x values as there are acceptable wage offers, and as many NLF states as there are acceptable wage offers. As will be seen below, this imposes as many restrictions on the distribution of non-market times, G(x), as there are acceptable wage offers.

The solution to (9) (for a discrete distribution F(.)) can be written formally as

$$(11) \quad p(t) = p(0)\exp\{Qt\}$$

where the matrix exponential is defined by

$$(12) \quad \exp\{A\} = \Sigma\, A^i / i!$$

(See Braun, 1978.)

Equation (11) gives the probability of being in different activities at any time in the future. For example, if a person was employed at time zero and there is a single acceptable wage, the probability of being in any activity in the future would be

$$p(t) = [1\ 0\ 0\ 0]\ \exp\{Qt\}$$

More importantly, since everyone follows equation (9) in his transitions between activities, equation (9) describes how the proportions of people in each activity changes. Equation (11) give the aggregate proportions of people in activities at t when the proportion at zero, p(0), is known. The steady state occurs when p'=0. The solution to pQ=0 gives the steady state.

The differential equations (8-10) hold for (w,z) fixed. The values n_1 and n_2, however, depend on (w,z) in the following way. The density of non-market (x) values in the population at time t is n(x,t). The n_1-types are those with x between x^*_0 and x^*_2: $\{x: x^*_0 < x < x^*_2\}$. Thus,

$$n_1(t) = \int_{x_0}^{x_2} n(x,t)dx\,.$$

The differential equation governing the population density n(x) (suppressing time) is

$$n'(x) = \mu g(x)[I\{x>\_x^*_2\}e + u] - [\mu + \lambda_2 pI\{x<x^*_2\}]n(x)$$

$$+ \ \mu g(x)\int_{x_0}^{\infty} n(x)dx$$

where $g(x)$ is the density of non-market value offers, and $I\{\cdot\}$ is the indicator function, i.e. 1 when the condition is true, and 0 when it is false. Integrating over n_1-types and n_2-types separately gives the differential equations (1c) and (1d). In other words for fixed (w,z) the differential equation governing the number of n_1-types depends only on the total number of n_1-types and n_2-types, and not their distribution. For given (w,z) this reduces the problem from infinite-dimensional (the integro-differential equation above together with analogous equations for e and u) to finite dimensional (the set of four simultaneous equations 8-10). Such a reduction is not possible when (w,z) vary, because the limits of integration, x*0 and x^*_2 (the individual's decision rules) depend on (w,z). When (w,z) change, the new values of $n_1(t)$ and $n_2(t)$ depend on the density n(x,t). When (w,z) changes, $n_1(t)$ and $n_2(t)$ are discontinuous.

IDENTIFICATION AND ESTIMATION

The identification of this model will be discussed only in a stationary environment. In other words, it will be assumed that z is fixed, and that the environment is stationary over time. Even for fixed z, identification and estimation of this model is delicate. First to deal with identification; i.e. given perfect data, can one hope to recover the parameters? (For expositional purposes I will focus on the case of a single acceptable wage. The arguments, however, hold for a general discrete wage distribution.) If the elements of the matrix Q above could be directly observed, the model would be over-identified; there are 12 independent transition rates but only 6 parameters (δ, μ, q_0, q_2, $\lambda_3 p$, $\lambda_2 p$). This requires the observation of the current wage and the value of non-market time. In fact, the elements cannot be observed for two reasons. First, one often does not have the exact time of transition, which is what one requires for directly estimating the elements of Q. More usually, one has a panel of people with their state at the beginning and end of a fixed time period, but no information on intermediate transitions. Second, the two NLF states (n_1 and n_2) cannot be separated - only their sum $n=n_1+n_2$ is observed. This introduces unobserved heterogeneity (generated by the model) into the transition process.

Say that one did in fact have continuous observations, but still could not distinguish n_1-types from n_2-types. In this case the observed transition matrix is Q*, given by (13)

(13)
$$\begin{array}{c|ccc|} & e & u & n \\ \hline e & -(\delta+uq_2) & \delta & \mu q_2 \\ u & \lambda_3 p & -(\lambda_3 p+\mu(1-q_0)) & \mu(1-q_0) \\ n & (n_1/n)\lambda_2 p & \mu q_0 & -((n_1/n)\lambda_2 p+uq_0) \end{array}$$

Note that this matrix is definitely not Markovian, because it includes the ratio n_1/n. Nonetheless, a number of parameters can be identified directly from the matrix Q*: δ, $\lambda_3 p$, μ, qo, q_2. The only parameter that cannot be identified directly is $\lambda_2 p$. (This seems to apply to the situation where only panel data are available also.) In steady state, however, n_1 and n_2 are known functions of the parameters, and so in the steady state $\lambda_2 p$ can be solved for.

Usually a transition matrix over a finite period of time, say one month, is observed. If one could separate n_1 and n_2, the observed monthly transition matrix would be

$$(14) \qquad \underline{P}(1) = [\underline{p}_{ij}]$$

where $\underline{p}_{ij}$ = (number who started in i and ended in j) / (number who started in j).

This corresponds to the finite period transition matrix generated by the intensity matrix Q:

$$(15) \qquad P(1) = P(0)\exp\{Q\}$$

where $P(0) = I$, the identity matrix.

The essence of the problem is calculating the matrix logarithm:

$$\underline{P}(1) = P(1) = e^{Q}$$

$$Q = \ln \underline{P}(1) \, .$$

This is not as trivial as might appear, because the logarithm of a matrix may give multiple, even complex roots; the elements of Q obviously should not be complex. Singer and Spilerman (1976a, 1976b) give specific conditions under which the equation $\underline{P}(1)=e^{Q}$ is invertible. Specifically, sufficient conditions are that the eigenvalues γ_i of $\underline{P}(1)$ must satisfy $0<\gamma_i\leq 1$, they must be real and distinct, and $\underline{P}(1)$ must satisfy $|\underline{P}(1)|>0$. In this case

$$Q = [\underline{P}(1)-I] - [\underline{P}(1)-I]^2/2 + [\underline{P}(1)-I]^3/3 - ...$$

The full problem is discussed by Singer and Spilerman. The present problem is somewhat more difficult, however, since the full matrix $\underline{P}(1)$ is not actually observed. In general the sufficient conditions quoted above (or the necessary conditions given in Singer and Spilerman) may not be satisfied. In that case there is no unique real matrix Q which satisfies (15). If there is no Q matrix which satisfies (15), then the model, although theoretically identified, is being confronted with data with which it is inconsistent. For a wage distribution with finitely many acceptable wages, the argument above carries through with no change. The wage is a state variable, and must be observed together with the value of non-market time. Note, however, that the mass of the wage distribution below the minimum acceptable wage (1-p in the case of a single wage) cannot be identified except by assuming a functional form for the wage distribution. In Flinn and Heckman (1982) or Heckman and Singer (1984b), this is referred to as a "recoverability" condition. The shape of the wage distribution above the minimum acceptable wage can be estimated non-parametrically, but 1-p, the mass below the minimum acceptable wage, can only be estimated by assuming a functional form.

Usually one does not observe the value of non-market time, and so cannot separate n_1-types from n_2-types. In this case (with a single acceptable wage) the observed finite period transition matrix is

$$(16) \quad \underline{P}^*(1) = \begin{bmatrix} \underline{p}_{ee} & \underline{p}_{eu} & \underline{p}_{en} \\ \underline{p}_{ue} & \underline{p}_{uu} & \underline{p}_{un} \\ \underline{p}_{ne} & \underline{p}_{nu} & \underline{p}_{nn} \end{bmatrix}$$

with $\underline{p}_{ij}$ as above with the two activities n_1 and n_2 collapsed into n. The matrix P*(1) is generated by the equation

$$(17) \quad P^*(1) = P^*(0) \exp\{Q(A)\} B$$

where

$$P^*(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & n_1(0)/n(0) & n_2(0)/n(0) \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$n_1(0)$ = proportion of people in NLF 1 at time zero (unobserved)
$n_2(0)$ = proportion of people in NLF 2 at time zero (unobserved)
n(0) = proportion of people in NLF at time zero (observed).

The problem is to solve for Q from $\underline{P}^*(1)$. In the steady state, $n_1(0)$ and $n_2(0)$ are constant, known functions of the parameters $A=(\delta, \mu, qo, q_2, \lambda_2 p, \lambda_3 p)$. (Specifically, $n_2=q_2$, $n_1=n-q_2$.). When these are substituted into P*(0), then P* can be written strictly as a function of the parameters A: P*(1;A). The parameters A can be adjusted until the calculated P*(1;A) equals the observed $\underline{P}^*(1)$; in other words A can be solved by solving for the zero of $\underline{P}^*(1)$-P*(1;A). (For finitely many acceptable wages the argument easily carries through.) It does not seem possible, a priori, to guarantee that a unique solution exists. For observed gross-flow data that I have examined, however, unique solutions do seem to exist.

As an alternative to assuming that the system is in steady state (for the case of a single acceptable wage), one can identify the model conditional on a value of $\pi(0)=n_2(0)/n(0)$, the number of n_2-types as a proportion of the number NLF. In either case, identification obtains because of the non-linear restrictions across the elements of Q.[5]

The worker's optimization problem imposes extensive non-linear restrictions across the elements of the matrix Q. Examining (10) shows that the form of the worker's optimization problem imposes non-linear restrictions across elements of the transition matrix. These restrictions are relatively easy to incorporate; the restrictions can be expressed analytically. In addition to those apparent in (10), however, there are additional restrictions imposed by the solution of the worker's problem. Imposing these requires the solution of a functional fixed point problem. As is apparent in Rust's (1984) discussion of this type of problem, this makes estimation considerably more difficult, both econometrically and computationally. In this model the problem simplifies in two important ways. Research is currently being undertaken to asses whether these simplifications carry over to other models

The probabilities q_0 and q_2 are not actually parameters, but rather depend on the solution of the worker's problem:

$$q_0 = P[X \leq x^*_0]$$

$$q_2 = P[X > x^*_2]$$

The underlying distribution of non-market times, $G(x)$, the cost of search, c, and the interest rate r are exogenous to the supply decision. Given the distribution $G(x;\eta)$ and values of c and r, the worker's optimization problem can be solved to give q_0 and q_2 as functions of the parameters:

$$q_0 = q_0(\delta,\mu,\lambda_2 p,\lambda_3 p,c,r,\eta)$$

$$q_2 = q_2(\delta,\mu,\lambda_2 p,\lambda_3 p,c,r,\eta)$$

The first simplification is to separate the estimation into a two-stage procedure. First the arrival rates and probabilities, $(\delta, \mu, q_0, q_2, \lambda_2 p, \lambda_3 p)$ are estimated from the data by maximum likelihood. Since this only involves incorporating the constraints apparent in the matrix (10), it is relatively easy. Second, the parameters of the distribution of non-market times, η, are solved for. For a two-parameter distribution $G(.;\eta)$, this involves no further restrictions: There are two estimated probabilities, q_0 and q_2, and two parameters, η. For a one-parameter family of distributions, there would be two estimates. Alternatively, a minimum distance estimator could be used, e.g.

$$\text{Min}_\eta\ [\underline{q}_0-q_0(\eta)]2 + [\underline{q}_2-q_2(\eta)]^2$$

The second simplification is that what initially appears to be a functional fixed point problem actually simplifies to two simultaneous non-linear equations. Initial inspection of the worker's optimization problem (equation 4) indicates that it is a functional fixed point problem. Rewriting (4) after the simplifying assumptions (linear utility, a single wage, z fixed, k set to zero), gives:

$$(4') \qquad V(w,x) = \text{Max}[V_1(w),V_2(x),V_3] = (TV)(w,x)$$

$$V_1(w) = (\mu+\delta+r)^{-1}[w + \mu EV(w,X) + \delta V(0,0)]$$

$$V_2(x) = (\mu+\lambda_2 p+r)^{-1}[x + \mu EV(0,X) + \lambda_2 pV(w,x)]$$

$$V_3 = (\mu+\lambda_3 p+r)^{-1}[-c + \mu EV(0,X) + \lambda_3 pV(w,0)] \ .$$

The variable w can take on the two possible values 0 (probability 1-p) or w^* (probability p). The variable x, however, is continuous. In this case, equation (4') appears to define a fixed point problem in an uncountably infinite number of dimensions. In fact, because of the reservation value property, only the reservation values need be determined. By direct manipulation when $\ln X \sim N(a,b)$ (see appendix B), the problem is reduced to a set of simultaneous non-linear equations. The dimensionality of the problem is reduced from uncountably infinite to two or three.

If the dimensionality of the parameter space of $G(.)$ (i.e. c and r plus the length of η) is smaller than the number of estimated probabilities q_i, then the restrictions imposed by the solution of the workers problem could be imposed in the first stage of the estimation. In this model, however, it is inadvisable to do so. The distribution $G(x;\eta)$ and the values c and r are not directly observed, can never be, and should probably be as loosely parameterized as possible. All that can be inferred directly from the data are the probabilities q_i. The safest procedure, then, is to allow q_i to be determined from the data (as parameters) and then, say, estimate the parameters η conditional on c, r, and the estimated values of q_i.

The asymptotic standard errors of η can be determined from the asymptotic standard errors of q_0 and q_2 by the delta method; see Billingsley, 1979, p.320. This would require solution of the functional fixed point problem, as well as calculation of the second derivatives of the fixed point operator itself. The fixed point problem would only have to be solved once, however, not at each iteration of the likelihood function as in Rust's problem. Even if the parametric form chosen for $G(x;\eta)$ were not the true functional form (as would probably be the case since the value of non-market time, x, cannot be observed) $A = (\delta, \mu, \lambda_3 p, \lambda_2 p, q_0, q_2)$ would still be consistently estimated. In other words, some of the parameters of interest can be consistently estimated by a computationally simple method even in the presence of (model-generated) heterogeneity. Estimation of the parameters in the first stage is relatively straightforward, given that the model is identified. For a single acceptable wage, the probability that an individual is in employment, unemployment, or NLF at t=1, $p^*(1)$, conditional on his initial probability density, $p^*(0)$, is given by an equation similar to (17):

$$p^*(1) = p^*(0) \exp\{Q\}B \qquad (18)$$

with B as defined in (17). (Note that $p^*(t)$ is a row vector, not a matrix as $P^*(t)$ is in (17).) For individuals in e and u, the initial probability densities $p^*(0)$ are

$$[\ 1 \quad 0 \quad 0 \quad 0\ ] \qquad\qquad [\ 0 \quad 1 \quad 0 \quad 0\ ]$$

respectively, since one observes the state at $t=0$. For individuals in NLF, $p^*(0)$ is

$$[\ 0 \quad 0 \quad 1-\pi_2(0) \quad \pi_2(0)\ ]$$

where $\pi_2(0) = n_2(0)/n(0)$, which depends on the unobserved $n_2(0)$.

If N people start in e and end in e, then the contribution of these people to the likelihood is (again with a single acceptable wage)

$$\Pi_1^N([1\ 0\ 0\ 0] \exp\{Q\} [1\ 0\ 0\ 0]') = \Pi_1^N p_{ee}$$

taking the notation from (14). If N people start in e and end in NLF, then the contribution to the likelihood is

$$\Pi_1^N([1\ 0\ 0\ 0] \exp\{Q\} [0\ 0\ 1\ 1]') = \Pi_1^N(p_{en1} + p_{en2})$$

If N people start in NLF and end in e, then the contribution to the likelihood is

$$\Pi_1^N([0\ 0\ 1\text{-}\pi_2(0)\ \pi_2(0)] \exp\{Q\} [1\ 0\ 0\ 0]')$$

$$= \Pi_1^N[p_{n1e}(1-\pi_2(0)) + p_{n2e}\pi_2(0)]$$

In summary, the log likelihood, considering all groups, is[6]

$$(19) \qquad £ = N_{ee}\ln p_{ee} + N_{eu}\ln p_{eu} + N_{en}\ln(p_{en1} + p_{en2})$$

$$+ N_{ue}\ln p_{ue} + N_{uu}\ln p_{uu} + N_{un}\ln(p_{un1} + p_{un2})$$

$$+ N_{ne}\ln[(1-\pi_2(0))p_{n1e} + \pi_2(0)p_{n2e}]$$

$$+ N_{nu}\ln[(1-\pi_2(0))p_{n1u} + \pi_2(0)p_{n2u}]$$

$$+ N_{nn}\ln[(1-\pi_2(0))(p_{n1n1} + p_{n1n2}) + \pi_2(0)(p_{n2n1} + p_{n2n2})]$$

For multi-wave panels, the log-likelihood function is just the sum of terms as above, with the N_{ij} and $\pi_2(.)$ receiving time subscripts. With multiple waves, it is in fact possible to estimate the initial proportion of n_2-types, $\pi_2(0)$, as a parameter. This is possible because one can impose the restriction that the implied proportion at the end of wave one be the initial proportion at the beginning of wave two, etc.

The present discussion can usefully be contrasted with that in Rust (1984). Rust works with a general class of models, and shows that the parameters can be consistently estimated. Rust, however, imposes assumptions sufficient to insure that the unobserved heterogeneity enters in a particular, tractable, form (assumption A6, p. 11). His assumption basically insures that the conditional probability of the unobserved state changing depends only on observed state variables. In the context of this model it would require that the probability of transiting from either n_1 or n_2 to a job would be the same, which violates the basic economic distinction between the two NLF states. (This could be imposed by assuming that job and NLF offers arrived simultaneously, so that the NLF state prior to an arrival would have no effect on the decision to take a job.) Interesting heterogeneity, generated by the structure of the model, seems to require dropping Rust's assumption A6, thus going outside Rust's structure. The two-stage estimation procedure, together with the reservation wage property of the optimization problem, leads to considerable simplification in estimation.

For a single wave of a panel, solving the zero of $\underline{P}^*(1)-P^*(1;A)$ provides maximum likelihood estimates. This can be seen by replacing the p_{ij} in (19) (generated by (15)) with p_{ij} generated by (17). In other words the elements of the 4x4 matrix $P(1)$ are replaced by the elements of the 3x3 matrix $P^*(1)$, to give a likelihood:

$$(20) \qquad £ = N_{ee}\ln p_{ee} + N_{eu}\ln p_{eu} + N_{en}\ln p_{en}$$

$$+ N_{ue}\ln p_{ue} + N_{uu}\ln p_{uu} + N_{un}\ln p_{un}$$

$$+ N_{ne}\ln p_{ne} + N_{nu}\ln p_{nu} + N_{nn}\ln p_{nn} \;.$$

The maximum likelihood estimates of p_{ij} are the $\underline{p}_{ij}$ from (16) (see Billingsley, 1961). This is the maximum value of the likelihood function with no constraints. The likelihood function is thus maximized with respect to A when $\underline{P}^*(1)=P^*(1;A)$ (if such a solution exists). For multiple waves, it should be clear that the maximum likelihood estimates, $p\hat{}_{ij}$, of the p_{ij} from (20), will not be the same as any of the monthly proportions, $\underline{p}_{ij}$. As pointed out above, the model is not identified with data from a single wave, except conditional on some assumption about $\pi(0)$, the initial number of n_2-types as a proportion of all those NLF. It appears, however, that it is only the parameter $\lambda_2 p$ that is not identified. This is certainly the case when one has spell length data. With spell length data, it is possible to consistently estimate the elements of the matrix Q^*, given in (13). From Q^* it is possible to identify all the parameters except $\lambda_2 p$.

Direct calculations indicate that with a single wave from a panel, $\lambda_2 p$ is not identified, but the other parameters are. Using data for January 1977 (described below) the model was estimated for various values of $\pi(0)$ between 0.80 and 0.20. The maximum spread (maximum minus minimum) for any of the five parameters (δ, μ, $\lambda_3 p$, qo, q_2), (as a percent of the mean of the parameter) was only 1.9%. This is probably within the numerical error of the optimization routine. The spread for $\lambda_2 p$, on the other hand, was 145% of the mean. In other words, only the hiring rate for those not in the labor force depends on the proportion of n_2-types. This is logical, since only n_1-types take jobs if they are offered. If the proportion of n_1-types is incorrectly assumed too low, then the estimated $\lambda_2 p$ (the instantaneous transition rate into employment conditional on being an n_1-type) will be estimated as too high. It appears that the parameters (δ, μ, $\lambda_3 p$, q_0, q_2) may be identified, while $\lambda_2 p$ and $\pi(0)$ are not.

PARAMETER ESTIMATES

This section of the paper provides estimates of the model when there is a single acceptable wage. The data used are gross flows from the current population survey. There are two data sets, corresponding to two attempts to adjust for basic problems in the data. The first source is the result of a research program undertaken by the Urban Institute during the early 1970's, and summarized in Holt et al. (1975). This provides data by demographic group (age, sex, and race). The monthly data have apparently been misplaced, but published averages for 1967-73 are available in Marston (1976). The second source is the result of recent research by Abowd and Zellner (1985). This provides data for the population 16+, and for males and females separately.

The original source for both data sets is the Current Population Survey (CPS) gross flow data.[2] Every month since 1949 the Bureau of the Census, under contract from the Bureau of Labor Statistics, has tabulated a variety of gross labor force flow measures. These are based on matched responses from common rotation groups across one or more months. The CPS consists of eight rotation groups, approximately 6,000 to 7,000 households each, or a total of 48,000 to 56,000 households per month. Each group is surveyed for four consecutive months, removed from the survey for eight months, and then surveyed for an additional four months. This means that at any given survey date, up to 75% of the respondents were also in the sample the previous month (those with current month in sample = 2, 3, 4, 6, 7, and 8). (In practice, some proportion of those eligible for matching are either missing entirely or have labor force status missing. More on this below.) From these one can estimate the number starting in one activity (say employment) last month and ending in another activity (say unemployment) this month. These are precisely the numbers N_{ij} of the previous section. One can also estimate the probability of starting in one state and ending in another, i.e. the $\underline{p}_{ij}$ of equation (16).

There are some problems with the CPS gross flow data. First, as mentioned above, there are records that cannot be matched. For any particular month about 15% of the eligible observations have labor force status missing from either this month or the previous month. Since there is reason to believe that these missing observations are not random, these introduce biases into the gross flow measures. One indication of the degree of the bias is that the marginal distributions of labor force status constructed from the gross flow often differ substantially from the full CPS proportions. A second problem is that random classification errors may bias the gross flows, even if they do not bias measurement of the levels. If respondents randomly change their reported status (without changing their true status) the flows will be biased even though the levels are not.

Abowd and Zellner undertake to adjust the reported raw gross flow data. Their methodology and results are reported in full in Abowd and Zellner (1985). They develop an adjustment procedure which

1. addresses the missing classification problem without assuming that the missing

[2] This and the following two paragraphs paraphrase Abowd and Zellner (1985).

labor force status information is missing at random, and

2. adjusts the resulting flow estimates for classification error.

Their method has some reasonable features:

1. it makes use of the information contained in the partially classified observations (those people who are observed this month but not last month or vice versa).

2. it uses summary data in a manner which permits direct applications to existing gross flow data. In other words their technique can be applied by other researchers to existing, published, flow data.

3. the technique is invertible (provided certain information is disclosed). That is, the original data can be recovered to allow others to apply their own adjustment procedures if Abowd and Zellner's is not appropriate for a particular application.

Abowd and Zellner provide some evidence and further arguments in favor of their technique. For the current application, the adjusted data seems adequate. There is the problem, however, that the data they report are population weighted; each person counts for about 1,000 members of the population. This means that all standard errors reported from maximum likelihood estimation must be adjusted by a factor of about 1,000.

Marston (1976) presents monthly transition matrixes by demographic group averaged over the period 1967 to 1973. I have used these as if they were generated by a single panel of the model outlined above. This is obviously not ideal. Marston's data, however, is the only source available by age, sex, and race. The advantages of using the data, flawed though it is, outweighs the disadvantages. The instantaneous transition matrix for the system is defined in (10). The six parameters ($\delta,\mu,q_0,q_2,\lambda_3 p,\lambda_2 p$) can be calculated as described above, assuming one is in the steady state, by solving for the zero of equation (17). Standard errors are not given for the estimates because the proportions matrix used is the average of actual monthly flow matrixes.

The parameter estimates shown in table 1 are generally what one would expect. Women and young people have much higher arrival rates of NLF values, and have much higher probabilities of quitting (q_2). As one would expect, women generally have higher mean distributions of NLF values than do men. Interestingly, the hiring rate from unemployment is pretty much the same for everyone, while the hiring rate from NLF is lower for older people. In other words, demographic groups do not differ dramatically with respect to going from unemployment to employment, but do differ dramatically with respect to leaving employment or going in and out of the labor force. Note that the model does reproduce the average levels of unemployment rather well (see table 3).

TABLE 1

CALCULATED PARAMETERS FOR SIMPLE SEARCH MODEL
CPS GROSS FLOW DATA AVERAGED OVER 1967-1973

							NLF Dist'n	
	delta	mu	Q_0	Q_2	$\lambda_3 p$	$\lambda_2 p$	μ	σ
White Males								
16-19	.0583	.7093	.1566	.1810	.4627	.2980	-.400	1.024
20-24	.0349	.4301	.2335	.0844	.4803	.4809	-	-
25-59	.0115	.2028	.2680	.0155	.4701	.1081	-.084	.226
White Females								
16-19	.0407	.6586	.1386	.2482	.5165	.1853	-.044	.719
20-24	.0223	.3017	.1688	.1824	.5923	.0796	.058	.301
25-59	.0128	.4586	.0407	.1034	.4066	.0529	.078	.300

NOTE: The parameters by demographic group use data from Marston (1976). The raw data are averages, over the period 1967-1973, of monthly flow transition matrixes, by demographic group. The parameters are:

delta: rate of arrival of layoffs in the job state. mu: rate of arrival of new NLF offers.
Q_0: probability of non-market benefits below lower reservation value (decision rule).
Q_2: probability of non-market benefits above upper reservation value (decision rule).
$\lambda_3 p$: rate of arrival of new jobs (product of search intensity and probability of finding vacant job in unemployment (search) activity. $\lambda_2 p$: rate of arrival of new jobs in NLF. NLF Dist'n: The distribution of NL value is assumed log-normal. The values assumed for cost of search and interest rate are c=0 and r=.005 per month. The mean and variance of the distribution G(x) are estimated conditional on the values of c and r.

TABLE 2
AVERAGE MONTHLY TRANSITION PROBABILITY BY DEMOGRAPHIC GROUP, 1967-1973, FROM MARSTON (1976)

	eu	en	ue	un	ne	nu
White Males						
16-19	0.0374	0.1205	0.3016	0.3295	0.1543	0.0625
20-24	0.0245	0.0381	0.3623	0.1791	0.1949	0.0610
25-59	0.0086	0.0037	0.3546	0.1023	0.0795	0.0382
White Females						
16-19	0.0272	0.1486	0.3065	0.3373	0.1002	0.0519
20-24	0.0155	0.0537	0.3903	0.1718	0.0533	0.0331
25-59	0.0087	0.0476	0.2733	0.2939	0.0432	0.0124

TABLE 3
OBSERVED AND STEADY STATE PROPORTION OF PEOPLE EMPLOYED, UNEMPLOYED, AND NLF

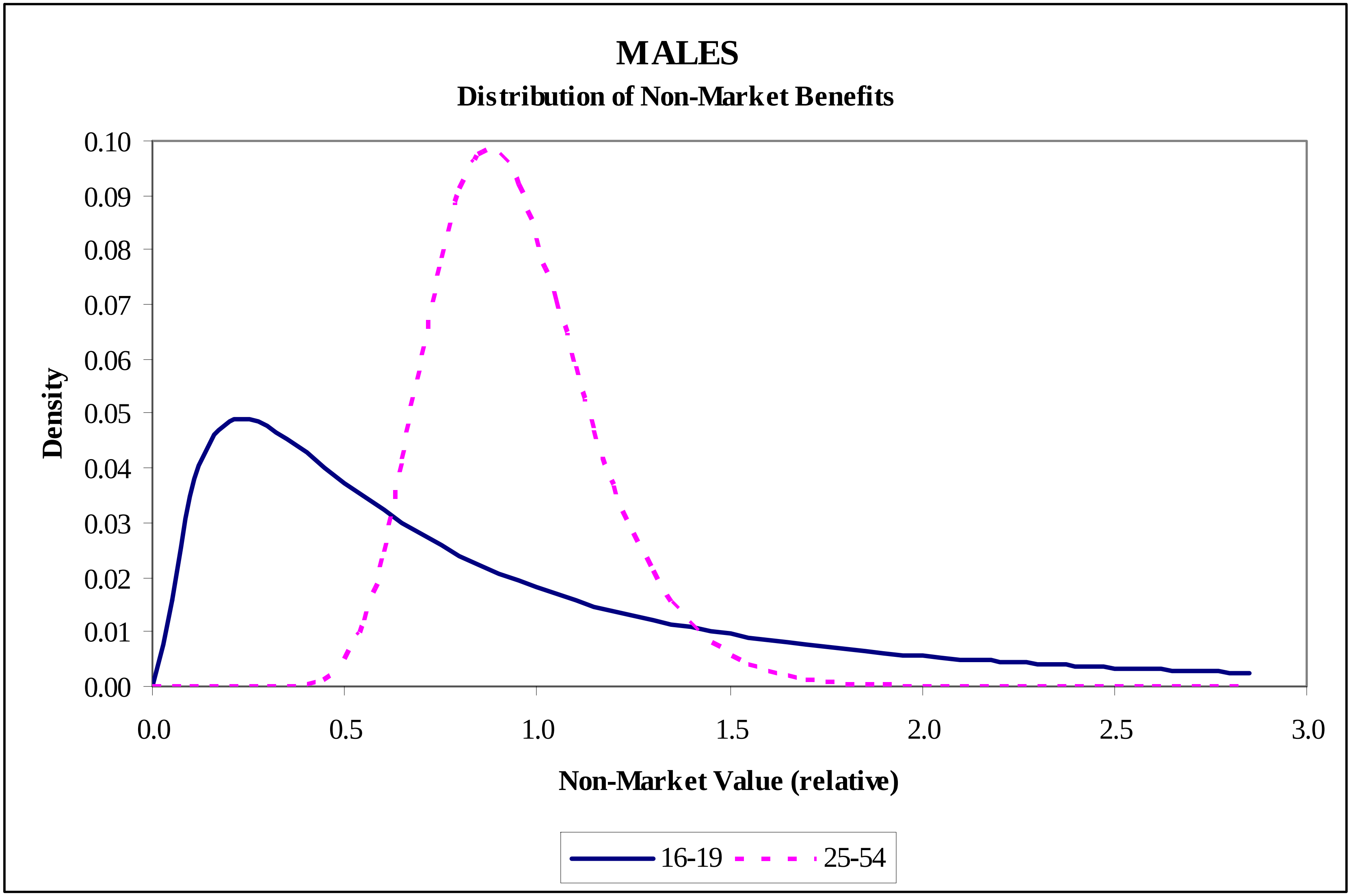
	OBSERVED[a]			STEADY STATE[b]		
	e	u	n	e	u	n
White Males						
16-19	50.96	7.22	41.82	52.77	7.10	40.13
20-24	78.14	5.61	16.26	79.16	5.33	15.51
25-54	94.35	2.18	3.46	93.02	2.15	4.83
White Females						
16-19	39.78	5.97	54.25	40.80	6.01	53.19
20-24	53.36	3.97	42.68	54.37	3.95	41.67
25-54	46.61	1.76	51.63	47.69	1.84	50.47

a The observed proportions of people is calculated from the average number of people in employment, unemployment, and NLF, 1967 to 1973 (civilian population only)

b The steady state proportions are calculated from the parameter estimate given in table 1 or alternatively the finite period transition matrixes from the CPS gross flow data, given in Marston (1976).

Figures 1 and 2 show the distributions of non-market times for males and females from the second stage estimation (the mean and variance of the distribution of non-market times are estimated by solving the fixed point problem). The distribution of non-market time is assumed to be log-normal. The mean and variance are calculated conditional on the values of c and r, and are as shown in table 1. The figures show that the distributions of non-market benefits (relative to the wage) are substantially different between men and women and between teenagers and older workers. (White males 20-24 is not shown because the data are inconsistent with the model.) Interestingly, white females 20-24 and 25-59 have distributions of non-market benefits (relative to wages) almost identical. Behavior of the two groups, however, is different in important respects. The average participation rate for females 20-24 is 57.3%, while it is only 48.4% for females 25-54. (Steady state rates implied by the model are 58.3% for 20-24, 49.5% for 25-59. See table 3.) Since both female groups have essentially the same distribution of non-market benefits, the difference must be the results of differences in other parameters.

To determine which parameter accounts for most of the difference between the two groups, the model for females 25-59 was solved, but with female 20-24 parameters sequentially substituted. The first row of table 4 shows the steady state values of e and u for females implied by the estimated parameters. The second row shows the values for e and u when the arrival rate of layoffs (δ) is changed to that of females 20-24 (all others at their estimated values). The third row shows the values when δ reverts to the estimated value, but the arrival rate of non-market offers (μ) is changed to that of females 20-24 (all others at their estimated values). The fourth row shows that the arrival rate of jobs in unemployment makes the difference, and it is the only parameter that makes a substantial difference to the employment ratio. The faster arrival of jobs makes unemployment more valuable to younger women, and induces a higher unemployment ratio (number unemployed / population). The higher pool of unemployed workers and higher conditional probability of finding a job when unemployed means there are more younger women working. This happens even though younger women have both higher layoff rates (δ, 0.022 versus 0.013) and higher quit rates (uq_2, 0.055 versus 0.047). Note that a higher arrival rate of jobs is unambiguously good for a worker, even though it may raise the steady-state unemployment ratio.


MALES
Distribution of Non-Market Benefits
Density
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Non-Market Value (relative)
16-19
25-54

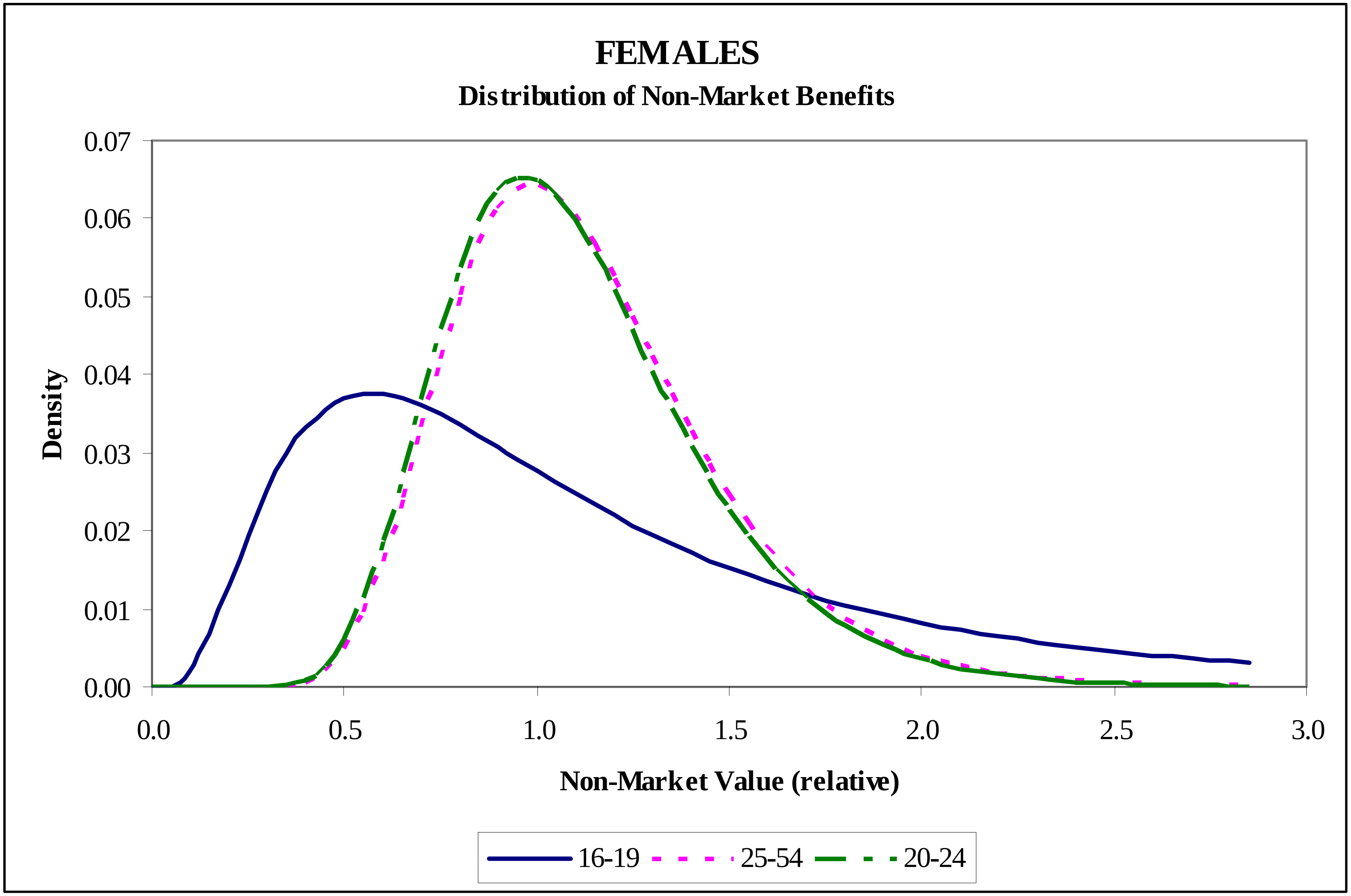

FEMALES
Distribution of Non-Market Benefits
Density
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Non-Market Value (relative)
16-19
25-54
20-24

TABLE 4
COMPARISON OF STEADY STATES
ACROSS DEMOGRAPHIC GROUPS
AMOUNT ATTRIBUTABLE TO SPECIFIC PARAMETERS[a]

	e	u	e	u
	WHITE FEMALES 25-59			
Estimated	47.66	1.84	47.66	1.84
After substituting parameters from				
	Females 20-24		Males 25-59	
δ	38.91	1.96	49.10	1.80
μ	48.70	2.89	50.37	3.55
λ_3	55.23	3.54	50.61	2.52
λ_2	47.38	1.14	48.14	0.85
G(x) (dist'n)	-	-	94.99	2.69
	WHITE FEMALES 20-24			
Estimated	54.39	3.96	-	-
			WHITE MALES 25-59	
Estimated	-	-	93.05	2.15

a The row labeled "Estimated" are the steady state values implied by the model from table 3 The rows labeled "After substituting... " refer to the steady state implied by substituting for the white female 25-59 value of the specified parameter the value for the demographic group in the column heading.

A similar analysis is done to compare males and females 25-59. This shows that virtually all the difference in the employment ratio between males and females 25-59 is the result of different NLF opportunities relative to wages. Since parameters for each demographic group are estimated separately, and since the value function is homogeneous of degree one with respect to the value of w, NLF values are all relative to the wage. The wage used in estimation is the same for all demographic groups (and was chosen as 1.183). The NLF distributions are thus comparable across demographic groups, but in a relative and not an absolute sense. The mean of the white male 25-59 distribution is 0.943, while that of the white female is 1.131. In other words, the mean NLF offer for women is higher, relative to their offered wage, than for men. The offered wage for men may be higher than for women, so that the absolute means of the NLF offers may not be very different. In any case, when the female 25-59 optimization problem is solved using the NLF distribution for males 25-59, the proportions employed and unemployed are close to that for men 25-59. Changing the hiring and firing rates has virtually no effect on the proportion employed. The arrival of non-market offers, however, does affect the proportion unemployed.

The differences between demographic groups are dramatic. Analyzing the mean length in each activity gives a perspective on the differences. For a Markov process, the mean length in a state is the inverse of the exit rate; if the diagonal element of the instantaneous transition matrix is $q_{ii}<0$, then the mean duration is $-1/q_{ii}$. The exit rates for employment and unemployment can be read directly from the transition matrix (equation 10), and the implied mean durations are shown in table 5. NLF is not a state of the Markov transition process, while n_1 and n_2 are. The mean duration for n_1 and n_2 are calculated as $-1/q_{ii}$. Since the exit rates from n_1 and n_2 are not the same, the hazard for exit from NLF is not constant. The problem is similar to, but more difficult than, the standard mixture of Markovs problem. In this case an n_1-type may change into an n_2-type, and vice-versa, before exiting to employment or unemployment. The density of leaving times and the mean duration can be solved for (see Cox and Miller, 1965, section 4.6).

TABLE 5
AVERAGE DURATION (MONTHS)
IMPLIED BY ESTIMATED MODEL PARAMETERS
(UNDER ASSUMPTION OF STEADY STATE)

	Employment	Unempl't	N1	N_2	Not in LF
White M					
16-19	5.4	0.94	1.9	1.7	3.6
20-24	14.0	1.2	1.6	2.5	3.2
25-59	34.9	1.6	6.0	5.0	7.8
White F 16-19	4.9	0.92	2.3	2.0	5.3
20-24	12.9	1.2	5.4	4.1	10.5
25-59	16.6	1.2	8.4	2.4	16.5

Average duration (in months) implied by the estimates in table 1. Calculated as described in the text.

As one would expect, the mean stay in NLF differs between males and females, with the mean duration for males lower than for females at each age category. The mean duration in NLF is over twice as long for females as for males age 25-59. Much of this difference arises because of the high transition rate from n_1 to unemployment for men, and the high transition rates between n_1 and n_2 for women. The implied mean duration of a job varies considerable, both across ages and between sexes. The low is 4.9 months for female teenagers and the high is 34.9 months for males 25-59 (a difference of 7 times). In contrast, the mean duration of unemployment only varies from 0.92 for female teenagers to 1.6 for males 25-59. The major differences between demographic groups appear in their employment (and NLF) rather than their unemployment behavior. This is consistent with Coleman (1985b), which argues that heterogeneity in not-unemployment is important in accounting for observations from the March CPS work experience data.

Differences between demographic groups also appear in their responses to changes in the underlying parameters such as hiring rates or layoff rates. In particular, I examine the effect on steady-state employment and unemployment of changes in the layoff rate and the hiring rate. There are two effects of a change in an exogenous parameter: first the direct effect on the steady state resulting from a different exogenous transition rate, and second the changes in the worker's decision rules which changes endogenous transition rates. The direct effect is the change in the solution to the steady state equation pQ=0 (see equation 9) when the decision rules do not change: $\partial e/\partial\delta$. The total effect is the change in the steady state when the endogenous decision rules change:

$$\partial e/\partial\delta + (\partial e/\partial q_0)dq_0/d\delta + (\partial e/\partial q_2)dq_2/d\delta \ .$$

The direct change in the steady state with respect to a parameter a is found by substituting $n_1=1-e-\mu-n_2$ into pQ=0. One then obtains a set of three simultaneous equations of the form

$$A + Bz = 0$$

with

$$z^T = [e\ u\ n_2] \ .$$

$$A = \begin{bmatrix} \lambda_2 p \\ \mu q_0 \\ \mu q_2 \end{bmatrix}$$

$$B = \begin{bmatrix} -(\delta+\mu q_2+\lambda_2 p) & \lambda_3 p-\lambda_2 p & -\lambda_2 p \\ \delta-\mu qo & -(\lambda_3 p+\mu) & 0 \\ 0 & 0 & -\mu \end{bmatrix}$$

Taking the derivative with respect to α gives

$$dz/d\alpha = -B\text{-}1[\partial A/\partial\alpha + (\partial B/\partial\alpha)z] \ .$$

The derivatives of the decision rules involve solving the contraction mapping that defines the optimization problem, and solving for the derivatives of the reservation wages. (These will depend on the assumed functional form for the distribution of the unobserved value of non-market time, G(.).) This is done in appendix B.

Table 6 shows the effect on employment and unemployment of changes in three parameters: the layoff rate (δ) the hiring rate from unemployment ($\lambda_3 p$), and the wage (w).

TABLE 6
EFFECTS ON EMPLOYMENT AND UNEMPLOYMENT
RESULTING FROM A 1% CHANGE IN PARAMETERS

		CHANGE IN					
		δ		$\lambda_3 p$		w	
		EFFECT ON					
		e	u	e	u	e	u
White Males							
16-19	direct	-0.054	0.029	0.057	-0.031	0.0	0.0
	total	-0.091	0.024	0.098	0.126	0.527	0.071
25-59	direct	-0.034	0.018	0.032	-0.017	0.0	0.0
	total	-0.049	0.018	0.015	-0.003	0.706	0.006
White Females							
16-19	direct	-0.030	0.015	0.055	-0.029	0.0	0.0
	total	-0.071	0.010	0.127	0.023	0.912	0.121
20-24	direct	-0.030	0.015	0.058	-0.028	0.0	0.0
	total	-0.117	0.010	0.365	0.062	2.975	0.142
25-59	direct	-0.031	0.007	0.038	-0.009	0.0	0.0
	total	-0.136	0.003	0.185	0.044	5.329	0.224

The direct effect is $\delta[\partial e/\partial \delta]$.
The total effect is $\delta[\partial e/\partial \delta + (\partial e/\partial q_0)dq_0/d\delta + (\partial e/\partial q_2)dq_2/d\delta]$.

The main point in table 6 is that different demographic groups have quite different responses to parametric changes. For example, the effect on employment of a change in the hiring rate from unemployment is almost 5 times larger for females 25-59 than for males (0.185 versus 0.039). The difference is mostly the result of different effects on decision rules. For both groups the direct effect, $\lambda_3 p(\partial e/\partial \lambda_3 p)$, is about the same (0.038 for females, 0.032 for males). The change in e resulting from a change in q_0, $(\partial e/\partial q_0)dq_0/d\lambda_3 p$, however, is larger for females (0.409 versus 0.083).

Interestingly, for both males and females 25-59 the indirect effect through changes in quit rates, $(\partial e/\partial q_2)dq_2/d\lambda_3 p$, is negative (-0.047 for females, -0.069 for males). In other words, increased hirings from unemployment lead to higher quit rates. At first glance this seems anomalous, but it is not. An increase in the hiring rate from unemployment directly increases the value of unemployment. Since movement from NLF to unemployment is more likely than movement from employment to unemployment ($\mu q_0 > \delta$ for all groups), the value of NLF increases more than the value of employment. This implies that the reservation NLF value, x^*_2, falls.[3]

TABLE 7
EFFECT ON VALUE FUNCTION
RESULTING FROM A 1% CHANGE IN PARAMETERS

	CHANGE IN δ		$\lambda_3 p$		w	
	EFFECT ON					
	Ve	Vu	Ve	Vu	Ve	Vu
White Males						
16-19	-9.014	-8.868	9.458	10.08	126.1	124.1
25-59	-4.739	-4.661	4.409	6.144	220.7	217.0
White Females						
16-19	-4.486	-4.391	8.114	8.739	98.58	96.50
20-24	-3.852	-3.768	10.21	11.77	131.9	129.0
25-59	-2.314	-2.200	2.677	3.552	118.0	112.1

The first entry (-9.014) is $\delta[\partial V_e/\partial \delta]$.

From table 6 it is clear that changes in the wage generate large changes in steady state employment. Changes in the wage, however, also generate large changes in the worker's value function. Table 7 shows the changes in the value of employment and unemployment associated with a one percent change in the parameter. A one percent change in the wage has a substantially larger effect than a one percent change in layoffs or hiring rates. The reason for this is that one can think of the change in layoffs or hiring rates as only a transitory effect; once hired the value of the hiring rate is not too important. In contrast, changes in the wage have a direct and lasting impact on the value of all states, since the wage provides utility directly. Tables 6 and 7 imply that, given the choice, workers may prefer to realize an exogenous decrease in steady-state employment through a rise in the layoff rate or a fall in the hiring rate rather than a change in the wage. For prime age white males, the fall in the value of being employed (for each 0.01 fall in steady-state employment-to-population ratio) is .97 if layoffs rise, 3.1 if wages

[3] It should be remembered, however, that the changes contemplated in table 5 are pure comparative statics comparisons. In general equilibrium, changes in a parameter exogenous to the worker's decision problem (but endogenous to the joint worker-firm problem) may be associated with changes in other exogenous parameters. In addition, the dynamics associated with changes in parameters may be of as much or more importance than the changes in the steady state. (See Coleman, 1985a, for estimates of the total population in a non-stationary setting.) Nonetheless, the differences between older men and women, and between the effect of changes in transition rates versus changes in wages, stand out.

fall.[4] Table 8 shows the effect on the value function when the given parameter changes so that steady-state employment-to-population ratio falls by 0.01.

TABLE 8

EFFECT ON VALUE FUNCTION RESULTING
FROM A 0.01 FALL IN EMPLOYMENT-TO-POPULATION RATIO

	CHANGE IN					
	δ		$\lambda_3 p$		w	
	EFFECT ON					
	Ve	Vu	Ve	Vu	Ve	Vu
White Males						
16-19	0.990	3.695	0.965	0.800	2.393	17.48
25-59	0.967	2.589	1.131	2.119	3.126	361.7
White Females						
16-19	0.632	4.391	0.639	3.800	1.081	7.975
20-24	0.329	3.769	0.197	0.849	0.443	9.085
25-59	0.170	7.333	0.145	0.807	0.221	5.004

The first entry (0.990) is $.01*(\partial V_e/\partial\delta)/[\partial e/\partial\delta + (\partial e/\partial q_0)dq_0/d\delta + (\partial e/\partial q_2)dq_2/d\delta]$

In a general equilibrium version of this model, layoff rates and hiring rates (in addition to the wage) would equilibrate supply and demand. Given that workers' utility may be very sensitive to changes in the wage, it would be reasonable to see large changes in layoff and hiring rates (i.e. direct changes in the levels of employment and unemployment without changes in the wage) in response to shifts in productivity.

Table 9 shows the results of estimating the model by maximum likelihood using monthly flow data over the period January 1977 to December 1982 (72 months). The asymptotic standard errors are only an approximation. The reason is that the flows reported by Abowd and Zellner (1985) are estimates of the U.S. civilian population from

---

4

$dV_e/d\delta$ = -4.739

$de/d\delta$ = -0.049

$(dV_e/d\delta)_\delta$ = $(dV_e/d\delta):(de/d\delta)$ = -4.739/-0.049

= 96.71 per 1 unit change in e resulting from a change in δ

= 0.9671 per 0.01 unit change in e.

dV_e/dw = 220.7

de/dw = 0.706

$(dV_e/dw)w$ = $(dV_e/dw):(de/dw)$ = 220.7:0.706

= 312.6 per 1 unit change in e resulting from a change in w

= 3.126 per 0.01 unit change in e.

the CPS sample. Individuals in the CPS sample are weighted by their proportional representation in the population to arrive at population estimates. In addition, Abowd and Zellner adjust the data (as outlined above). The weighting in the sample is very approximately 1,000. This means that the value of the likelihood function must be multiplied by approximately 1,000. Because of the very small standard errors, the results will not be changed substantially if the weighting factor were different by an order of magnitude. Nonetheless, the standard errors must be interpreted with some caution.

TABLE 9

PARAMETER ESTIMATES FOR CIVILIAN POPULATION 16+
USING MONTHLY GROSS FLOW DATA, 1977 TO 1982

			Parameters				
	μ	δ	$\lambda_3 p$	$\lambda_2 p$	q_0	q_2	p_0
	.199941	.0171804	.268225	.0318715	.151250	.0987056	.209437
			Asymptotic Standard Errors				
	649.7E-6	55.46E-6	756.8E-6	114.8E-6	18.84E-6	14.01E-6	12.16E-3
				Inverse Hessian x10-5			
42.208E-3							
334.37E-6	307.59E-6						
427.29E-6	498.66E-6	57.272E-3					
-1.4506E-3	-22.606E-6	-1.0398E-6	1.3171E-3				
-24.111E-3	-250.02E-6	3.2871E-3	176.82E-6	35.506E-6			
-22.230E-3	-292.69E-6	11.355E-6	1.3666E-6	13.211E-3	19.616E-6		
-55.871E-3	-3.0503E-3	-39.398E-3	40.103E-6	49.222E-3	36.534E-3	14.798	

CONCLUSION

The introduction to this paper pointed out two substantive theoretical differences between this model and other models of aggregate labor supply. First, this model presumes an optimization problem for the individual, and then aggregates individual behavior to derive the laws of motion for the economy as a whole. A representative agent approach does not work. Individual choices are discrete, whereas the aggregate shows more or less continuous variation in employment and unemployment; it is impossible for the aggregate economy to behave like an individual worker. Even though the laws of motion for the aggregate imply continuous variation, they are derived by modeling the discrete choice problem and then aggregating, rather than assuming workers make marginal decisions. In addition to the discrete nature of the individual decisions, different demographic groups show distinctly different behavior.

Second, technological and informational constraints, which make it impossible to find a job instantly, enter both the worker's value function and the aggregate laws of motion. These constraints, taken as exogenous by the worker, may act as "prices" that equilibrate supply and demand in a general equilibrium model. Firms would have costs of hiring and firing workers at high rates. In competition, these hiring and firing rates will change until the marginal benefit to a worker of an increase equals the marginal cost to the firm. A shock to the worker's labor supply function or the firm's labor demand function will be equilibrated by changes in all "prices," including the wage and hiring rates. In a general equilibrium model, then, one might see large fluctuations in employment associated with small fluctuations in real wages. This is consistent with the statistically insignificant estimates of aggregate labor supply elasticities often found, e.g. in Altonji and Ashenfelter (1980), or Altonji (1982):

> For most specifications, the current real wage, the expected real wage, and the expected real rate of interest are either insignificantly related to unemployment and labor supply, or have the wrong sign. [Altonji, p. 784]

It is also consistent with the general feeling that wages fluctuate too little to be explained by a model where wages equilibrate labor supply and demand:

> In the equilibrium model described in sections 2 and 3, efficient employment fluctuations are achieved by equating both the marginal product of labor and the marginal rate of substitution (between leisure and consumption) to the real wage rate, in each period. Under reasonable assumptions on the supply and demand parameters, this implies that the real wage should fluctuate more than employment. … In the U.S. manufacturing data used in section 6 below, employment has substantially larger variance than the real wage. [Kennan, 1983, section 5.]

The empirical estimates of this paper are for a steady state version of the model using data by demographic group from Marston (1976). The results, broadly, are threefold. First, the difference in aggregate employment and unemployment between prime age males and females seems to be primarily the result of differences in the distribution of non-market opportunities, rather than differences in hiring or layoff rates. Second, demographic groups show marked differences in their employment and NLF behavior, but not so much difference in their unemployment to employment transition of their mean duration of unemployment. This is consistent with the arguments in Coleman (1985b) that heterogeneity in not-unemployment is quite important in accounting for certain unemployment statistics. Finally, across all demographic groups, the value of employment falls more in response to changes in wages than to changes in either layoffs or hiring rates (when each changes so that the employment-to-population ratio falls by 0.01). This is interesting when taken in conjunction with the observation that, for some industries at least, wages fluctuate much less than employment. It could be optimal that employment is directly equilibrated by changes in layoff or hiring rates.   Further work with this type of discrete choice model of the labor market is necessary. Non-stationarity (changes in offer distributions and hiring and layoff rates) must be introduced. Equilibrium between firms and workers must be studied. Nonetheless, this type of model seems both feasible and fruitful for analyzing the aggregate labor market.

APPENDIX A

EXISTENCE OF THE VALUE FUNCTION

Blackwell's characterization of the contraction mapping theorem is used to prove that V(w,x,k,z) exists, is continuous, and increasing in (w,x). First, the equations defining V(w,x,k,z), 1-4, are rewritten in the form of an operator T on an arbitrary function v(w,x,k,z):

(1)

$$(Tv)(w,x,k,z) = \mathrm{Max}\{ \ (\mu+\delta+\nu+\lambda_1+r)^{-1}[u(w)+\mu Ev(w,X,0,z)+\delta v(0,0,k,z) + \lambda_1 Ev(\max(w,W),0,0,z) + \nu Ev(w,0,0,Z)],$$

$$(\mu+\lambda_2+\nu+r)^{-1}[u(x)+\mu Ev(0,X,0,z)+\lambda_2 Ev(W,x,0,z)+\nu Ev(0,x,0,Z)],$$

$$(\mu+\lambda_3+\nu+r)^{-1}[u(k-c)+\mu Ev(0,X,k,z)+\lambda_3 Ev(W,0,k,z)+\nu Ev(0,0,k,Z)] \ \}$$

This operator takes continuous functions over $[0,\infty]x[0,\infty]x\{0,k\}$ into continuous functions.

Some conditions are required on u(.), the arrival rates μ, δ, λ_i, ν, and the distributions of W, X, and Z. The random variables Z, W, X are independent random functions over the positive real line $\Re^+$ and a probability space $(\Omega,\Im,P)$. In other words, for $\omega\in\Omega$

$$Z = Z(z,\omega)$$

$$W = W(w,z,\omega)$$

$$X = X(x,z,\omega)$$

To maintain the interpretation of z as a measure of the strength of the economy I require that $W(w,z,\omega)$ and $X(x,z,\omega)$ are non-decreasing in z for each ω, w, and x. This insures that the probability of receiving a wage (or non-wage) offer at least as high as some w' is non-decreasing:

$$P[\omega : W(w,z,\omega) \geq w'] = 1 - F_{w,z}(x)$$

is non-decreasing in z, for a given w. Since wage offers are non-negative this means the mean of W(.), for a given w, will increase with z:

$$\int_\Omega W(w,z_1,\omega)dP(\omega) \leq \int_\Omega W(w,z_2,\omega)dP(\omega) \qquad \text{for } z_1 \leq z_2.$$

To maintain non-decreasingness of the value function in the wage and non-wage offers, $W(w,z,\omega)$ and $X(x,z,\omega)$ are assumed non-decreasing in w and x, respectively. This has intuitive appeal if wages reflect productivity, and current wage offers serves as signals of productivity. Note, however, that the existence of an optimal policy and of the value function does not require this assumption (nor the assumption above about W(.) and X(.) non-decreasing in z). It is made to insure the value function increases in w and x.

Further restrictions on the utility function and the random variables are:

$$\lim_{x\to\infty} u(x) < \infty$$

$$E|W|,\ E|X|,\ E|Z| < \infty \quad \text{for all } w, x, z\ .$$

In addition, all arrival rates are finite for all z.

Using the sup-norm to define the distance between two functions v^0 and v^1:

(2) $$d(v^0,v^1) = \| v^0 - v^1 \| = \sup_{\substack{w,x,z\geq 0 \\ k'=k \text{ or } 0}} |v^0(w,x,k',z) - v^1(w,x,k',z)|$$

the space of continuous, bounded functions is closed (all Cauchy sequences converge), and one can use Banach's fixed point theorem (see Wouk, 1979, p. 21):

BANACH'S FIXED POINT THEOREM

If

(3) $$d(Tv^0,Tv^1) \leq \beta d(v^0,v^1) \qquad \text{for } 0<\beta<1 \text{ and any } v^0,\ v^1$$

then the functional fixed point equation $v=Tv$ has a unique solution. The solution is v^*, with

(4) $$T^i v^0 \to v^* \text{ as } i \to \infty \text{ for any continuous, bounded } v^0$$

where $T^2v = T(Tv)$, etc.

Rather than using Banach's theorem directly, it is often easier to use

BLACKWELL'S CONDITIONS:

If an operator, T, from the space of bounded continuous functions to the space of bounded functions satisfies

a) monotonicity: $v^0 < v^1 \Rightarrow Tv^0 \leq Tv^1$

b) $T(v^0+k) < Tv^0 + \beta k$, $0 < \beta < 1$, k constant.

Then T is a contraction mapping with

$$d(Tv^0,Tv^1) \leq \beta d(v^0,v^1) \quad 0<\beta<1 \text{ .}$$

PROOF

$$v^0(w,x,k,z) \leq v^1(w,x,k,z) + \| v^0 - v^1 \| \quad \text{by definition of sup-norm}$$

$$(Tv^0)(w,x,k,z) \leq (Tv^1)(w,x,k,z) + \beta v^1 \quad \text{by a), b)}$$

$$v^1(w,x,k,z) \leq v^0(w,x,k,z) + \| v^0 - v^1\| \quad \text{by definition of sup-norm}$$

$$(Tv^1)(w,x,k,z) \leq (Tv^0)(w,x,k,z) + \beta\| v^0 - v^1 \| \quad \text{by a), b).}$$

Using these two inequalities gives

$$| (Tv^0)(w,x,k,z) - (Tv^1)(w,x,k,z) | < \beta\| v^0 - v^1 \| \quad \text{and so}$$

$$\| Tv^0 - Tv^1 \| \leq \beta\| v^0 - v^1 \|$$

$$d(Tv^0,Tv^1) \leq \beta d(v^0,v^1) \text{ .}$$

Q.E.D.

For the operator T in (1), conditions a) and b) are easy to verify. T is non-decreasing in v, so a) is satisfied.

$$[T(v+c)](w,x,k,z) < (Tv)(w,x,k,z) + \beta c$$

where $\beta = \max \{ (\mu+\delta+\lambda_1+\nu)/(\mu+\delta+\lambda_1+\nu+r), (\mu+\lambda_2+\nu)/(\mu+\lambda_2 p+\nu+r),$

$(\mu+\lambda_3 p+\nu)/(\mu+\lambda_3 p+\nu+r) \}$.

As long as $r>0$, $0<\beta<1$, and so condition b) is satisfied. Thus

$$V(w,x,k,z) = (TV)(w,x,k,z)$$

has a unique solution, V(w,x,k,z). Further, V(w,x,k,z) is non-decreasing in (w,x). The operator T (see 1) takes functions that are non-decreasing into functions that are non-decreasing; it preserves the non-decreasing property. Since V(w,x,k,z) is the limit of $(T^i v^0)(w,x,k,z)$, V(w,x,k,z) will be non-decreasing if $v^0(w,x,k,z)$ is non-decreasing. Since $v^0(w,x,k,z)=0$ is a valid starting function, and is non-decreasing, V(w,x,k,z) is non-decreasing in (w,x).

Note that the restriction $\lim_{x\to\infty} xu(x)<\infty$ means the above proof does not hold for u(x) = x, i.e. wealth maximization, unless the range of W and X are restricted to be finite. The problem is that V(w,x,k,z) is not bounded. To circumvent this problem, define

$$V^*(w,x,k,z) = V(w,x,k,z) - \mathrm{Max}[w/(\delta+r),\ x/(\mu+r)]$$

$$= \mathrm{Max}[V_1(w,z),V_2(x,z),V_3(k,z)] - \mathrm{Max}[w/(\delta+r),\ x/(\mu+r)]$$

When the random variables W, X, Z have finite means, then the operator defined by $(Tv)(w,x,k,z)-\mathrm{Max}[w/(\delta+r),\ x/(\mu+r)]$ (with T from (1) above) takes positive, bounded, continuous functions into positive, bounded, continuous functions. Thus the operator defined by

$$(Sv)(w,x,k,z) = (Tv)(w,x,k,z) - \mathrm{Max}[w/(\delta+r),\ x/(\mu+r)]$$

with $u(x)=x$ has a unique solution $V^*(w,x,k,z)$. This proves that $V(w,x,k,z) = V^*(w,x,k,z) + \mathrm{Max}[w/(\delta+r),x/(\mu+r)]$ exists and is unique. In addition one can show that it is increasing in (w,x) for fixed z.

To prove the assertion, take the operator T, and replace every occurrence of $v(w,x,z)$ with $v(w,x,z) = v^*(w,x,z) + \mathrm{Max}[w/(\delta+r),x/(\mu+r)]$. The expression for V_1 then becomes

$$(\mu+\delta+\nu+\lambda_1+r)^{-1}\{w+\mu Ev^*(w,X,0,z) + \delta v^*(0,0,k,z)$$
$$+ \lambda_1 Ev^*(\max(w,W),0,0,z) + \nu Ev^*(w,0,0,Z)\}$$
$$+ (\mu+\delta+\nu+\lambda_1+r)^{-1}\{w + \mu awP[X<aw/b] + \mu b\int_{aw/b}^{\infty} XdP(X) + \lambda_1 awP[W<w]$$
$$+ \lambda_1 a\int_{w}^{\infty} WdP(W) + \nu aw\}$$

where $a = 1/(\delta+r)$, $b = 1/(\mu+r)$. By judicious addition and subtraction, one can obtain

$$(\mu+\delta+\nu+\lambda_1+r)^{-1}\{ w+\mu Ev^*(w,X,0,z) + \delta v^*(0,0,k,z)$$
$$+ \lambda_1 Ev^*(\max(w,W),0,0,z) + \nu Ev^*(w,0,0,Z)$$
$$- \mu awP[X>aw/b] + \mu b\int_{aw/b}^{\infty} XdP(X) - \lambda_1 awP[W>w] + \lambda_1\int_{w}^{\infty} WdP(W)$$
$$- ((\mu+\delta+\nu+\lambda_1+r)/(\mu+r))\, I_{[x>aw/b]} \}$$
$$+ \mathrm{Max}[w/(\delta+r),x/(\mu+r)] \ .$$

The first term in brackets is bounded from above if $v^*(.)$ is bounded from above, provided that W and X have finite means. (For example, $wP[W>w] \leq \int_w^\infty W dP(W) < \infty$.) As x grows large the first term tends to $-\infty$, but as w grows large the first term is finite.

Similar formulae can be derived for V_2 and V_3. All are bounded from above. The corresponding first term for V_2 tends to $-\infty$ as $w \to \infty$, but is finite as $x \to \infty$. Thus $(Tv)(w,x,k,z)$ can be written as

$$(Tv)(w,x,k,z) = \mathrm{Max}[\, v^*_1(w,z), v^*_2(x,z), v^*_3(k,z) \,] + \mathrm{Max}[\, w/(\delta+r), x/(\mu+r) \,]$$

The first term on the right is bounded from above (because each of v^*_i are bounded from above) and bounded from below (because at least one of v^*_i are finite as either w or x $\to\infty$). Thus the operator

$$(Sv^*)(w,x,k,z) = \mathrm{Max}[\, v^*_1(w,z), v^*_2(x,z), v^*_3(k,z) \,]$$

takes continuous, bounded functions into continuous, bounded functions. Mortensen (1985) in an earlier (but independent) paper uses the same method for proving existence in a related class of problems.

APPENDIX B

SOLVING FOR THE VALUE FUNCTION

The value function is evaluated as follows. First, the value function is reproduced from the text:

$$V_1(w) = [w + \mu EV(w,X) + \delta V(0,0)]/(\mu+\delta+r)$$

$$V_2(x) = [x + \mu EV(0,X) + \lambda_2 pV(w,x)]/(\mu+\lambda_2 p+r)$$

$$V_3 = [-c + \mu EV(0,X) + \lambda_3 pV(w,0)]/(\mu+\lambda_3 p+r)$$

Using

$$\mu EV(w,X) = \mu(1-q_2)V_1(w) + \mu \int_{x_2}^{\infty} V(x)dG(x)$$

$$\delta V(0,0) = \delta V_3$$

etc.

this can be re-written as

$$(1) \qquad V_1(w) = [w+\mu(1-q_2)V_1(w) + \mu\int_{x_2}^{\infty} V_2(x)dG(x) + \delta V_3] / (\mu+\delta+r)$$

$$(2) \qquad V_2(x) = [x+\mu q_0 V_3 + \mu \int_{x_0}^{\infty} V_2(x)dG(x) + \lambda_2 p V_1(w)] / (\mu+\lambda_2 p+r) \quad \text{for } x \leq x_2^*$$

$$(3) \qquad V_2(x) = [x+\mu q_0 V_3 + \mu \int_{x_0}^{\infty} V_2(x)dG(x) + \lambda_2 p V_2(x)] / (\mu+\lambda_2 p+r) \quad \text{for } x \geq x_2^*$$

$$(4) \qquad V_3 = [\mu q_0 V_3 + \mu \int_{x_0}^{\infty} V_2(x)dG(x) + \lambda_3 p V_1(w)] / (\mu+\lambda_3 p+r)$$

Now, note that $V_2(x_0^*)=V_3$, and that $V_2(x_2^*)=V_1(w)$. In addition, write

$$V_{2l} = \int_{x_0}^{x_2} V_2(x)dG(x)$$

$$V_{2u} = \int_{x_2}^{\infty} V_2(x)dG(x)$$

Then, equations 1-4 (evaluating 2 at x_0^* and 3 at x_2^*) can be written as

$$(\mu q_2+\delta+r)V_1 \qquad\qquad - \mu V_{2u} \qquad - \delta V_3 \qquad = w$$

$$\lambda_2 p V_1 + \mu V_{2l} + \mu V_{2u} - [\mu(1-q_0)+\lambda_2 p+r]V_3 = -x^*_0$$

$$-(\mu+r)V_1 + \mu V_{2l} + \mu V_{2u} + \mu q_0 V_3 = -x^*_2$$

$$\lambda_3 p V_1 + \mu V_{2l} + \mu V_{2u} - [\mu(1-q_0)+\lambda_3 p+r]V_3 = c$$

The assumption is maintained that $\ln X \sim N(\nu,\sigma_2)$. If values for w, c, and r are assumed, then the solution of the above set of simultaneous equations (using the estimated values of μ, $\lambda_2 p$, $\lambda_3 p$, q_0, q_2, and δ) gives values for ν and σ. (The choice of a value for w is immaterial, since that only sets the nominal units of account.) Solving the above equations give the following relations between the unknown values of V_1, V_{2u}, V_{2l}, V_3, x^*_0, and x^*_2:

$$(5) \qquad x_0 = [\, x_2 (\lambda_3 p-\lambda_2 p) - c (\mu+\lambda_2 p+r) \,] / [\, \mu+\lambda_3 p+r \,]$$

$$(6) \qquad V_3 = V_1 - (x_2+c) / (\mu+\lambda_3 p+r)$$

$$(7) \qquad V_{2u} = [\, V_1 (\mu q_2+r) - w + \delta(x_2+c) / (\mu+\lambda_3 p+r) \,] / \mu$$

$$(8) \qquad V_{2l} = q_1 V_1 + [\, w + c - (x_2+c)(\delta+\mu(1-q_0)+\lambda_3 p+r) / (\mu+\lambda_3 p+r)] / \mu$$

Equations (2) and (3) give

$$V_2(x) = [\ x + \mu q_0 V_1 + \mu q_1 V_{2l} + \mu q_1 V_{2u} + \lambda_2 p V(w,x)\ ] / (\mu+\lambda_2 p+r)$$

Integrating separately over the ranges $[x^*_2, \infty)$, $[x^*_0, x^*_2]$ gives

$$(9) \qquad (\mu+r)V_{2u} = \int_{x_2}^{\infty} x\, dG(x) + \mu q_2\ [\ q_0 V_1 + q_1 V_{2l} + q_2 V_{2u}\ ]$$

$$(10) \qquad (\lambda_2 p+\mu+r)V_{2l} = \int_{x_0}^{x_2} x\, dG(x) + \mu q_1\ [\ q_0 V_1 + q_1 V_{2l} + q_2 V_{2u}\ ] + \lambda_2 p q_1 V_1(w)$$

By substituting (6-8) into (9), one arrives at the expression

$$(11) \qquad V_1(w)\ \{\ \mu q_2\ (1-q_2) - [\ \mu(1-q_2) + r\ ](\mu q_2+r) / \mu\ \} =$$

$$- \int_{x_2}^{\infty} x\, dG(x) + q_2\ [\ (x_2+c)(\mu+\lambda_3 p+r+\delta) / (\mu+\lambda_3 p+r) - (w+c)\ ]$$

$$- [\ \mu(1-q_2)+r\ ]\ [\ w - \delta(x_2+c) / (\mu+\lambda_3 p+r)\ ] / \mu$$

By substituting (6-8 and 11) into (10), one gets

$$0 = \int_{x_0}^{x_2} x\, dG(x) - w(\mu+\lambda_2 p+r) / \mu - c(\mu(1-q_1) + \lambda_2 p + r) / \mu$$

$$- q_1(x_2+c)(\mu q_0-\delta) / (\mu+\lambda_3 p+r)$$

$$- (x_2+c)[\ \mu(1-q_1) + \lambda_2 p + r\ ][\ \delta + \mu(1-q_0) + \lambda_3 p + r\ ] / [\ \mu(\mu+\lambda_3 p+r)\ ]$$

Writing $\eta_2=\mu+\lambda_2 p+r$, $\eta_3=\mu+\lambda_3 p+r$, this is

$$(12) \qquad 0 = - w\eta_2/\mu + \int_{x_0}^{x_2} x\, dG(x) - q_1(x_2+c)(\mu q_0-\delta) / \eta_3$$

$$- (q_1 - \eta_2/\mu)(x_2+c)\ [\ 1 + (\delta-\mu q_0) / \eta_3\ ] + c(q_1 - \eta_2/\mu)$$

Note that

$$q_0 = P[X < x_0] = \int_0^{x_0} dG(x)$$

$$q_1 = P[x_0 < X < x_2] = \int_{x_0}^{x_2} dG(x) = 1-q_0-q_2$$

$$q_2 = P[X > x_2] = \int_{x_2}^{\infty} dG(x)$$

I assume that $\ln X \sim N(\nu,\sigma_2)$, so that

$$g(x)\,dx = (2\pi\sigma^2 x^2)^{-1/2} \exp\{ [(\ln x - \nu)/\sigma]^2/2 \}$$

$$(13) \qquad \int_{x_0}^{x_2} x\,dG(x) = \exp\{ (\nu+\sigma) / (2\sigma) \} \{ \Phi[(\ln(x_2)-\nu-\sigma)/\sigma]-Q[(\ln(x_1)-\nu-\sigma)/\sigma]$$

$$(14) \qquad q0 = \int_{0}^{x_0} dG(x) = \Phi[(\ln(x_0)-\nu)/\sigma]$$

$$(15) \qquad q_2 = \int_{x_2}^{\infty} dG(x) = 1 - \Phi[(\ln(x_2)-\nu)/\sigma]$$

Values for $A=(\mu,q0,q_2,\lambda_2 p,\lambda_3 p,\delta)$ are estimated from the data. Values for r (r=0.005, or 6.2% per year) and c (c=0) are assumed. The four equations (12-15) form a system of non-linear, simultaneous equations. They can be solved by using Newton's method for finding the zero of a system of equations. The solution gives values for (x_2, ν, σ). From these one can use (5) and (10) to solve for V_1, and then (6-8) to solve for V_{2l}, V_{2u}, and V_3. The results are not very sensitive to the choice of values for r and c.

APPENDIX C

DERIVATIVES OF THE VALUE FUNCTION

Derivatives of the value function are evaluated as follows. First, the value function is reproduced from the text:

$$V_1(w) = [w + \mu EV(w,X) + \delta V(0,0)] / (\mu+\delta+r)$$

$$V_2(x) = [x + \mu EV(0,X) + \lambda_2 pV(w,x)] / (\mu+\lambda_2 p+r)$$

$$V_3 = [-c + \mu EV(0,X) + \lambda_3 pV(w,0)] / (\mu+\lambda_3 p+r)$$

Using

$$\mu EV(w,X) = \mu(1-q_2)V_1(w) + \mu \int_{x_2}^{\infty} V(x)dG(x)$$

$$\delta V(0,0) = \delta V_3 \text{ etc.}$$

this can be re-written as

$$(1) \qquad V_1(w) = [w+\mu(1-q_2)V_1(w) + \mu\int_{x_2}^{\infty} V_2(x)dG(x) + \delta V_3] / (\mu+\delta+r)$$

$$(2a) \qquad V_2(x) = [x+\mu q0V_3 + \mu\int_{x_0}^{\infty} V_2(x)dG(x) + \lambda_2 pV_1(w)] / (\mu+\lambda_2 p+r) \quad \text{for } x<x_2^*$$

$$(2b) \qquad V_2(x) = [x+\mu q0V_3 + \mu\int_{x_0}^{\infty} V_2(x)dG(x) + \lambda_2 pV_2(x)] / (\mu+\lambda_2 p+r) \quad \text{for } x>x_2^*$$

$$(3) \qquad V_3 = [\mu q0V_3 + \mu\int_{x_0}^{\infty} V_2(x)dG(x) + \lambda_3 pV_1(w)] / (\mu+\lambda_3 p+r)$$

Write $V_2'(x)$ and V_3' to represent either

$$dV_2(x)/d\delta \qquad \text{or} \qquad dV_2(x)/dw$$

$$dV_3/d\delta \qquad\qquad dV_3/dw$$

Equations (2) and (3) can be differentiated and written as

(4)

$$V_2'(x) = [\mu q0V_3' + \mu(\int_{x_0}^{x_2} V_2'(x)dG(x) + \int_{x_2}^{\infty} V_2'(x)dG(x)) + \lambda_2 pV_1'] / (\mu+\lambda_2 p+r) \quad \text{for } x<x_2$$

$$V_2'(x) = [\mu q0V_3' + \mu(\int_{x_0}^{x_2} V_2'(x)dG(x) + \int_{x_2}^{\infty} V_2'(x)dG(x)) + \lambda_2 pV_2'(x)] / (\mu+\lambda_2 p+r) \quad \text{for } x>x_2$$

$$V_3'(x) = [\mu q0V_3' + \mu(\int_{x_0}^{x_2} V_2'(x)dG(x) + \int_{x_2}^{\infty} V_2'(x)dG(x)) + \lambda_3 pV_1'] / (\mu+\lambda_3 p+r)$$

$V_2'(x)$ is constant for $x<x_2$ and $x>x_2$, and does not exist for $x=x_2$. (Fortunately, both left and right derivatives exist, but they are different.) Defining

$$V_{2l}' = V_2'(x) \text{ for } x < x_2$$

$$V_{2u}' = V_2'(x) \text{ for } x > x_2$$

the three equations (4) can be written as

$$[\ \mu(1-q_2) + r\ ]V_{2u}' - \mu q_1 V_{2l}' - \mu q0V_3' = 0$$

$$\mu q_2 V_{2u}' - [\ \mu(1-q_1) + \lambda_2 p + R\ ]V_{2l}' + \lambda_2 pV_1' = 0$$

$$\mu q_2 V_{2u} + \mu q_1 V_{2l}' - [\ \mu(1-q0) + \lambda_3 p + r] + \lambda_3 pV_1' = 0$$

The solutions to these three equations is

$$V_{2l}' = V_1' \,[\, (\mu+\lambda_3 p+r)\beta_2/\beta_1 - (\lambda_3 p-\lambda_2 p) \,] / [\mu+\lambda_2 p+r]$$

$$V_{2u}' = V_1' \,[\, (\mu+\lambda_3 p+r)\beta_2/\beta_1 - \lambda_3 p \,] / [\mu+r]$$

$$V_3' = V_1' \,\beta_2/\beta_1$$

$$\beta_1 = [\, \mu(1\text{-}q0)+\lambda_3 p+r \,] - [\, (\mu q_2)/(\mu+r) + (\mu q_1)/(\mu+\lambda_2 p+r) \,] / [\mu+\lambda_3 p+r]$$

$$\beta_2 = [\, \lambda_3 p-(\mu q_2\lambda_3 p)/(\mu+r) \,] - [\, \mu q_1(\lambda_3 p-\lambda_2 p) \,] / [\mu+\lambda_2 p+r]$$

To find $V_1'(w)$, differentiate (1), first by w, then by δ:

$$(dV_1/dw)(\mu+\delta+r) = 1 + \mu(1-q_2)(dV_1/dw) + \mu q_2(dV_{2u}/dw) + \delta(dV_3/dw)$$

$$(5) \qquad (dV_1/dw) = 1/\beta_3$$

$$\beta_3 = \mu \,\{\, q_2 1 - [\, (\mu+\lambda_3 p+r)\beta_2/\beta_1 - \lambda_3 p \,]/[\mu+r] \,\} + \delta[1 - \beta_2/\beta_1] + r$$

$$(dV_1/d\delta)(\mu+\delta+r) + V_1(w) \;=\; \mu(1-q_2)(dV_1/d\delta) + \mu q_2(dV_{2u}/d\delta) + \delta(dV_3/d\delta) + V_3$$

$$\Rightarrow\; (dV_1/d\delta) \;=\; [\, \mu q_2(dV_{2u}/d\delta) + \delta(dV_3/d\delta) + V_3 - V_1(w) \,] / (\mu q_2+\delta+r)$$

$$(6) \qquad (dV_1/d\delta) = [V_3 - V_1(w)] / \beta_3$$

REFERENCES

J.G. Altonji, (1982), "The intertemporal substitution model of labour market fluctuations: An empirical analysis," Review of Economic Studies, vol. 49, pp. 783-824.

J.G. Altonji and O. Ashenfelter, (1980), "Wage movements and the labour market equilibrium hypothesis," Economica, vol. 47, pp. 217-245.

P. Billingsley, (1979), Probability and Measure, John Wiley & Sons, New York M. Braun, (1978), Differential Equations and their Applications , 2nd ed., New York, Springer-Verlag 1978.

K. Burdett and D.T. Mortensen, (1978), "The effects of layoffs on optimal search strategies," ms. September 1978.

____________, (1978), "Labor supply under uncertainty," in Research in Labor Economics, vol. 2 R.G. Ehrenberg ed., JAI Press Inc., Greenwich CT.

T. Coleman, (1984), "Essays on aggregate labor market business cycle fluctuations," Ph.D. Dissertation, University of Chicago, December 1984.

T. Coleman, (1985), "Employment, hours, and non-marginal decisions in the U.S. labor market: Theory and evidence," Research Paper No. 273, State University of New York at Stony Brook, July 1985.

T. Coleman, (1985b), "How short is unemployment duration? Evidence from the CPS work experience survey reconsidered," Research Paper No. 270, State University of New York at Stony Brook, June 1985.

C. Flinn and J. Heckman, (1982), "New methods for analyzing structural models of labor force dynamics," Journal of Econometrics, 18, pp. 115-168.

R.E. Hall, (1980), "Labor supply and aggregate fluctuations," in On the State of Macro-Economics, ed. by K. Brunner and A.H. Meltzer. Amsterdam, North-Holland, 1980.

J.J. Heckman and B. Singer, (1984a), "A method for minimizing the impact of distributional assumptions in econometric models for duration data," Econometrica, vol. 52, no. 2, pp 271-320.

J.J. Heckman and B. Singer, (1984b), "Econometric duration analysis," Journal of Econometrics, vol. 24, pp. 63-132.

H.B. Kaitz, (1970), "Analyzing the length of spells of unemployment," Monthly Labor Review, vol. 93, pp. 11-20, November 1970.

J. Kennan, (1983), "An econometric analysis of equilibrium labor market fluctuations," University of Iowa College of Business Administration Working Paper Series No. 83-16. Presented at the summer 1983 Econometrics Society meetings.

J. Kennan, (1984) "Wage smoothing and labor market equilibrium: Exploratory data analysis," ms. date January 5, 1984, University of Iowa.

S.A. Lippman and J.J McCall, "The economics of job search," Economic Inquiry, 14, Part I (June 1967), Part II (September 1976).

S.T. Marston, (1976), "Employment instability and high unemployment," Brookings Papers on Economic Activity, no. 1.

D.T. Mortensen, "The existence of optimal quitting strategies for a class of job turnover models," R.R #85-06, C.V. Starr Center for Applied Economics, New York University.

S.J. Nickell, (1979), "Estimating the probability of leaving unemployment," Econometrica, vol. 47, pp. 1249-66.

J. Rust, (1984), "Maximum likelihood estimation of controlled discrete choice processes," ms. May 1984, Social Systems Research Institute, University of Wisconsin.

S.W. Salant, (1977), "Search theory and duration data: A theory of sorts," Quarterly Journal of Economics, vol. 91, pp. 39-57, February 1977.

U.S. Bureau of Labor Statistics, Handbook of Labor Statistics, Bulletin 2070.