ACCURATELY ESTIMATING AND BUILDING THE YIELD CURVE

*RISK* Yield Curve Course, October 1999

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ÆQUILIBRIUM INVESTMENTS LIMITED
Outline

• General Approach to Fitting Yield Curve
• Mathematics of Yield & Forward Curve
• Simple Example
• Use of and Criteria for Curves
• Choice of Input Data
• Various Functional Forms
Fitting The Yield Curve - Outline

• General Approach
  – Define discount function with a functional form for forward curve
  – Choosing market data (inputs) and appropriately describing the instruments
  – Define and implementing an appropriate objective function and fitting methodology
  – All instruments priced through discount function
Fitting the Yield Curve - Diagram

Discount Function

<table>
<thead>
<tr>
<th>Type</th>
<th>Matur</th>
<th>Freq</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swap</td>
<td>2yr</td>
<td>Semi</td>
<td>6.36</td>
</tr>
<tr>
<td>Swap</td>
<td>3yr</td>
<td>Semi</td>
<td>6.50</td>
</tr>
<tr>
<td>Swap</td>
<td>5yr</td>
<td>Semi</td>
<td>6.66</td>
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Objective Function and Fitting Method

Swaps

Options
Fitting - Implementation Issues

• Modularize
  – Separate curve from instrument details

• Re-use code
  – Use DiscFact in curve and swap pricing
  – Use same subroutines to price instruments

• Build in flexibility
  – Changing forward curve and instruments
Yield Curve Mathematics

• Term structure of interest rates expressed as
  – forward curve
  – zero curve
  – discount curve

• I like to use forward curve, but matter of taste
Yield Curve Mathematics - cont’d

- Discount curve in terms of zeros / forwards

\[ df(t) = e^{-y(t) \cdot t} \]

- Relation between forwards and zeros:

\[ y(t) = \left[ \int_0^t f(u) du \right] / t \]
Yield Curve Mathematics - cont’d

• These expressions are for continuously compounded forward and zero rates
• Zero is an “average” of forwards - smoothed
Example - Forward Curve

• Forward curve functional form
  – Piece-wise constant forwards $f_1$, $f_2$, $f_3$
  – Breaks at 2, 3, 5 years

• Discount factor function

$$df(t) = \exp[-f_1 * t] \quad \text{for } t \leq 2$$
$$df(t) = \exp[-2f_1 - f_2*(t-2)] \quad \text{for } 2 < t \leq 3$$
$$df(t) = \exp[-2f_1 - f_2 - f_3*(t-3)] \quad \text{for } 3 < t \leq 5$$
Example - Market data

• Just swaps for example

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• NPV of swap

\[
NPV = \sum_{i=1}^{2\text{ yrs}} \text{df}(i / 2) \cdot \text{rate} / 2 + 100 \cdot \text{df}(\text{yrs}) - 100
\]
Example - Objective Function

• Fit forwards so that all NPVs are zero
  \[ \text{NPV}(2\text{yr}; f_1, f_2, f_3) = 0 \]
  \[ \text{NPV}(3\text{yr}; f_1, f_2, f_3) = 0 \]
  \[ \text{NPV}(5\text{yr}; f_1, f_2, f_3) = 0 \]

• Fit forwards sequentially (bootstrap)
  – Fit \( f_1 \) by solving \( \text{NPV}(2\text{yr}; f_1) = 0 \) since 2 year swap depends only on first forward rate.
  – Then fit \( f_2 \) by solving \( \text{NPV}(3\text{yr}; f_1, f_2) = 0 \)
Example - Results

- First forward is 2 yr rate
  \[ 6.26\%_{cc} = 6.36\%_{sab} \]
- Second and third forwards easy to bootstrap
  - Can do it with HP12C
    \[ f_2 \quad 6.70\%_{cc} = 6.81\%_{sab} \]
    \[ f_3 \quad 6.83\%_{cc} = 6.94\%_{sab} \]
Example - Extensions

- Extending this is easy conceptually
- As always, devil is in the details
  - Swap payments may not fall on exact dates - e.g. holidays
  - Futures and deposits
- This general approach - separating forward curve from instrument - simplifies details
  - Details encapsulated in instrument subroutine
Curve Criteria

- **Mark-to-market (interpolator)**
  - Curve used for daily MTM of derivatives portfolio
  - E.g. swaps portfolio from liquid futures & swaps

- **Rich-cheap analysis (smoothing)**
  - Curve used to identify instruments whose market price is rich or cheap relative to others
  - E.g. US Treasury curve with 200 bonds
Curve Criteria - MTM

• Relatively few inputs, each reprices exactly
• Speed and simplicity
• Localization
• Reasonably smooth forwards
Curve Criteria - Rich/Cheap

- Generally many inputs, none fit exactly
- Smoothing noisy data to reasonable market curve
- Strong localization not required
- Speed and simplicity less important
Choice of Input Data

• Depends largely on use of the curve
• For MTM
  – Liquid instruments with good, easily observed market quotes
  – Instruments actually used to hedge the book
Input Data - Swaps MTM

• Generally three “sectors”
  – Money market - libor deposits
  – FRA / Futures
  – Swaps
Input Data - Deposits

• Generally needed to “tie-down” the front of the curve
  – In US I would use over night, 1 week, 2 week, 1 month, then switch to futures
  – Exact deposits used depends on futures dates

• But beware of liquidity problems with longer (e.g. 6 month) deposits
  – Longer deposits and shorter futures may not always match - choose liquidity?
Input Data - FRA / Futures

• Choose between FRAs and futures based on liquidity and transparency
  – In USD, CAD, GBP, EUR I would use futures.
  In some other currencies FRAs

• Big issue of convexity
  – FRA payoff is convex in rate
  – Futures payoff is linear in rate ($25 / tick)
Input Data - Futures Convexity

• Deposits, FRAs, swaps all same “class” of instrument
  – Can synthetically construct one from the other
  – Arbitrage
• Futures - with linear payoff - is different
• Futures is not simply PV off FRA curve
  – Must use term-structure model to price
  – There are approximations for the convexity correction
Input Data - Convexity Approximation

- Doust’s approximation for convexity correction:

\[ R_{\text{fut}} = \frac{R_{\text{fra}}}{DF_{\text{exp}}^{\frac{1}{2}}\sigma^2 t^*(t+\frac{1}{2})/(t+\frac{1}{4})} \]

- \( t \) = time to futures expiry (in years)
- \( R_{\text{fra}} \) = forward (FRA) rate from the curve
- \( DF_{\text{exp}} \) = discount rate to futures expiry date
- \( \sigma \) = volatility in decimal, i.e. 0.20.

Input Data - Swaps

- Instruments straightforward
  - But must get frequency, day-count, etc., correct
  - Details change between markets - advantage of separating curve and instruments

- Where to switch from futures to swaps
  - Depends on liquidity and hedge instruments
  - We used 4 years of futures, 5 year swap
Functional Forms

• Discuss three (four)
  – Piece-wise constant forward (PWCF)
  – Piece-wise linear zeros (PWLZ)
  – Piece-wise linear forwards (PWLF) - twisted and smoothed

• Do not discuss cubic splines
  – Popular, but problems with non-localization
  – In my opinion, not a good form for MTM
Functional Forms - PWCF

• Choose break points (usually instrument maturities)
• Forwards constant between breaks
Functional Forms - PWCF

Forward Rates

PWCF
Zero
Functional Forms - PWLZ

• Choose break points (usually instrument maturities)

• Zeros linear between breaks
  – Zeros linear, continuous across breaks (knots) but not smooth
  – Forwards linear between breaks, discontinuous across breaks

• Most common market method (or close)

• Large jumps in forwards
Functional Forms - PWLZ

Forward Rates

- Zero
- PWLZ

Coleman - Building the Yield Curve
Functional Forms - PWLF

• Choose break points (usually instrument maturities)

• Forwards linear between breaks
  – Generally more parameters than instruments
  – Twisted - set slope average of forwards on either side
  – Smoothed - minimize jumps and change in slopes
  – Two methods give virtually same results
Functional Forms - PWLF (twisted)
Risk and Hedging

• Risk measurement dependent on forward curve functional form
  – Constant forwards - risk interpolated approximately proportional to BPV
  – Linear zeros - risk interpolated approximately linearly

• Example - hedge 20 year with 10 & 30
  – Constant forwards - ratio of 22%/78%
  – Linear zero - ratio of 42%/58%
Addendum - Approximate Forwards

- Converts par yields (exact years) to PWCF
- Based on implicit function theorem
  - Par yield as function of forwards: \( y_i = Y(f_1,\ldots,f_i) \)
  - Implies \( dy_i = \sum_j a_{ij} df_j \quad a_{ij} = \frac{\partial y_i}{\partial f_j} \)
  - \( dY = A \cdot dF \quad dF = A^{-1} \cdot dY \)
  - Approx: \( F \approx A^{-1} \cdot Y \)
  - \( a_{ij} = \frac{\partial y_i}{\partial f_j} \approx \frac{\partial PV_i}{\partial f_j}/[dPV_i/dy_i] \)
    based on \( dPV_i = [dPV_i/dy_i]dy_i = \sum_j [\partial PV_i/\partial f_j]df_j \)
    \( \Rightarrow dy_i = \sum_j \{[\partial PV_i/\partial f_j]/[dPV_i/dy_i]\} \cdot df_j \)
    approximate \( [\partial PV_i/\partial f_j] \) by DV01 of forward bond
• Consider steeply down-ward sloping sterling swap curve, August 1999:

<table>
<thead>
<tr>
<th>Matur</th>
<th>Par Y</th>
<th>Approx</th>
<th>True PWCF</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6.74%</td>
<td>6.74%</td>
<td>6.74%</td>
</tr>
<tr>
<td>10</td>
<td>6.46%</td>
<td>6.08%</td>
<td>6.08%</td>
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<tr>
<td>20</td>
<td>5.98%</td>
<td>5.12%</td>
<td>5.14%</td>
</tr>
<tr>
<td>30</td>
<td>5.61%</td>
<td>3.91%</td>
<td>4.00%</td>
</tr>
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• Works well considering steepness of curve
  • Error 9bp at 20-30 years
Conclusion

• Fitting the yield curve not difficult
• Big returns to a methodical approach
• Choose forward curve functional form based on how curve is used
• Choice of functional form matters