Convexity And Correlation Effects in Swap Pricing

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Overall Themes

- Discounting Expected Cash Flows
- Use Simplest Model Which Solves Problem
- Focus on Hedging and Managing Risks

Correlation - Spread Options

- Products which naturally incorporate correlation
  - Single yield curve spreads
  - Across yield curves - swap spreads
  - Across yield curves - Gilts vs. Bunds
  - More complicated - index amortization swaps

Spread Option - Example

- Focus on specific example - call on 2 vs. 10 year swap yields
- Two contrasting views
- Two underliers: $Y_{10} - Y_2$
  - Correlation matters
- One underlier: S_{10vs2}
  - Correlation only enters indirectly

Spread Option - Pricing Theory

- Discount ECF using risk-neutral measure
- Take expectation of payout:

$$E[(Y_{10} - Y_2) - K \mid (Y_{10} - Y_2) > K]$$

or

$$E[S - K \mid S > K]$$

Spread Option - Pricing Theory

- Principle of using simplest model which fits
- Here we can use simple model
  - Option is European
  - Depends on two variables - Y_{10} & Y_2

Spread Option - Pricing Theory

- Now must make some decisions
- What are some reasonable assumptions about Y_{10}, Y_2, and S?
  - $\ln(Y_{10})$ & $\ln(Y_2)$ jointly normal with correlation ρ
  - S normally distributed with standard deviation s
- Introduce a new variable - ρ or s

Spread Option - Pricing Theory

- Focus on normal spread model

$$\text{NPV(call)} = \text{DF(3mth)} \int_{S=K}^{\infty} (S - K) g(S) dS$$

$$g(S) = \frac{1}{\sqrt{2\pi s^2 t}} \exp\left(\frac{(S - F)^2}{2 s^2 t} \right)$$

$$\text{NPV} = \text{DF(3mth)} s t^{.5} [\phi(d) + d\Phi(d)]$$

$$d = (F-K)/ s t^{.5}$$

Spread Option - Practical Pricing

- Pricing - must come up with inputs
  - F_{10} and F_2 off of forward curve (but see convexity adjustment below)
  - *s* - Standard Deviation of spread
- S.D. depends on correlation

$$Var(S) = Var(Y_{10}) - 2\rho SD(Y_{10})SD(Y_2) + Var(Y_2)$$

- Trading decision on level of *s* or ρ

Spread Option - Practical Pricing

- For US
  - 3 mth fwd 2 year rate 6.23
  - 3 mth fwd 10 year rate 6.96
  - Forward spread 73.8bp
  - LN volatility of 2 year 20.0%
  - LN volatility of 10 year 16.5%
  - Historical correlation 93%
  - Spread standard deviation 46 bp

Spread Option - Pricing and Risk

- Resulting Spread Option Price
- Three month call option, ATM
  - Price 9.1bp
  - Benefits if spread widens
- Three month put option, 10bp OTM
  - Price 5.0bp
  - Benefits if spread narrows

Spread Option - Hedging

- Must hedge against movements in either
  - yields, vols, correlations
  - Spread, standard deviation of spread
- Two ways of saying same thing

Spread Option - Hedging

- Difficult to hedge
- Flip side of difficulty in choosing level of *s* or ρ
- Usually must live with the correlation / spread standard deviation risk
- To hedge trade the spread - delta hedge

Spread Option - Risk Measurement

- Two ways of measuring risk for ATM
- Risks to spread and spread std. dev.

  1bp in spread 0.5bp   1bp in std. dev. 0.2bp
- Risks to yields, vols, and correlation

  1 bp in Y_2 0.5bp   1 bp in Y_{10} 0.5bp

  1 vol pt in Y_2 0.5bp   1 vol pt in Y_{10} 0.1bp

  1 percentage point in correlation 0.6bp

Spread Option - Hedging

- Either way, delta hedge spread
- When buy call, hedge by selling spread
- To sell yield spread buy 10 & sell 2 yr swap
  - Buy / sell 3mth forward swaps
  - Buy / sell DV01-weighted amounts
  - Call changes by 0.5bp for 1bp change in spread
  - If payout $10,000/bp, call changes by $5,000
  - Buy 7.1mm 10s, 27.4mm 2s

Model Choice - Simple vs. Complex

- Advantages of simple models
  - Easier to understand and implement
  - Focus directly on important aspects of problem (pricing & hedging correlation of Y_{10} and Y_2)
- Advantages of complex models
  - General model works in variety of applications (e.g. simple model might predict spreads widen without limit for long-dated options)
  - No need to force problem to fit model

Convexity - Adjusting Forward Rates

- Call as expectation of payout:

$$E[(Y_{10} - Y_2) - K \mid (Y_{10} - Y_2) > K]$$

- Focus on spread (underlier) itself

$$(Y_{10} - Y_2)$$

- Hedge spread by selling / buying bonds
- Spread is linear, hedge convex

Convexity - Adjusting Forward Rates

- Use risk neutral measure for which

$$E[\,PV(Y_{10})] = PV(Y_{10}^{f})$$
$$E[\,PV(Y_{2})] = PV(Y_{2}^{f})$$

- Find mean of distribution for which

$$\int_{0}^{\infty} PV(Y)g(Y;Y^{m},\sigma)dY = PV(Y^{f})$$

- This gives expectation of linear yield

$$E[(Y_{10} - Y_{2})] = (Y_{10}^{m} - Y_{2}^{m})$$

Convexity - Adjusting Forward Rates

- Call as expectation of payout:

$$E[(Y_{10} - Y_2) - K \mid (Y_{10} - Y_2) > K]$$

- PV of call is

$$NPV = DF(3mth)st^{.5}[\phi(d) + d\Phi(d)]$$

$$d = (F-K)/\, st^{.5}$$

- But F now uses adjusted means

$$F = (Y_{10}^{m} - Y_{2}^{m})$$

Convexity - Adjusting Forward Rates

- Adjustment is small, but can matter
- For 3 month ATM call
  - 2 year 6.225 forward, 6.230 adjusted
  - 10 year 6.964 forward, 6.979 adjusted
  - Spread 73.8 forward, 74.9 adjusted
  - Call 9.1bp off forwards, 9.7bp adjusted

Convexity - Calculating Adjustment

- Find mean of distribution for which

$$\int_0^\infty PV(Y)g(Y;Y^m,\sigma)dY = PV(Y^f)$$

- Various methods
  - Brute force numerical integration
  - Approximate integrand by piece-wise quadratic
  - More sophisticated approximations
- Brotherton-Ratcliffe & Iben
  - spread = $r^2 \sigma^2 T P'' / (2 P')$

Convexity - Calculating Adjustment

- Comparison of results
- 3 month forward 2 and 10 year swaps
  - Solving integral equation, 0.48bp, 1.55bp
  - B-R & I approximation, 0.48bp, 1.54bp
- 5 year forward 2 and 10 year swaps
  - Solving integral equation, 10.2bp, 31.7bp
  - B-R & I approximation, 9.5bp, 32.4bp

Convexity - Hedging

- Hedge spread as before
  - When buy call, sell spread
  - Sell spread in delta-weighted amount
  - Sell spread, buy 10s sell 2s in DV01-weighted amounts
- But now additional volatility risk
  - As volatility changes, convexity changes
  - Must hedge volatility exposure

Convexity - Libor-in-Arrears

- Same situation as for spread
- Libor-in-arrears payment is linear
  - Payment at t is L_t
  - Hedge is convex $FRA(L_t) = L_t / (1+L_t)$
- Find mean of distribution for which

$$\int_0^\infty FRA(L)g(L; L^m, \sigma)dL = FRA(L^f)$$

Convexity - Libor-in-Arrears

- Convexity effect is usually small
  - Libor usually quarterly
  - Swaps not too long
- Example
  - Quarterly libor, adjustment 0.5bp at 1yr, 2.4bp at 5yr
  - Annual libor, adjustment 1.8bp at 1yr, 9.2bp at 5yr

Conclusion

- Tried to show a few applications of
  - Discounting Expected Cash Flows
  - Use Simplest Model Which Solves Problem
  - Focus on Hedging and Managing Risks
- Managing derivatives risk
  - Choosing appropriate model & assumptions
  - Focus on hedging and managing risks
  - Not about high-powered mathematics

Problem 1 - Spread Option Price

Parameters for an ATM call spread option are as in the presentation:

3 mth fwd 2 year rate	6.23
3 mth fwd 10 year rate	6.96
Forward spread	73.8bp
Spread standard deviation	46 bp
Three-month discount factor	0.9865

1. Calculate the price of the option, using the formula given in the presentation and (if needed) the following approximation for $\Phi(d)$:

$$NPV = DF(3mth)\,s t^{.5}[\phi(d) + d\Phi(d)]$$

$$d = (F-K)/\,s t^{.5} \qquad \phi(d) = \exp(-d^2/2)/(2\pi)^{.5}$$

d<0

$$\Phi(d) = \exp\left[-\frac{(83|d|+351)|d|+562}{703/|d| + 165}\right]/2$$

d>0

$$\Phi(d) = 1 - \exp\left[-\frac{(83|d|+351)|d|+562}{703/|d| + 165}\right]/2$$

Problem 2 - Hedging Spread Option

What precisely is the hedge to the spread option?

Specifically, assume you bought an ATM call with payout $10,000/bp. In general to hedge a (bought) call you must sell the underlier. In this case, you must sell the 10yr minus 2yr yield spread. To sell the yield spread you must buy the 10 year and sell the 2 year forward swaps.

Answer the following specific questions:

1. What is the P&L on the option if the spread rises by 1bp?
2. What is the P&L on a $1mm position in the 10 year forward swap (with a DV01 of $701.70)? On the 2 year (with DV01 of $182.80)?
3. How much should you buy of the 10 year and sell of the 2 year?

Problem 3 - Spread on libor-in-arrears

What is the spread for a libor-in-arrears swap? Specifically, consider the in-arrears side of a 5 year swap against 1 year libor-in-arrears. The forwards and adjusted forwards are as below:

Yr of pmt	Fwd	Vol	Adj Fwd	Sprd	Disc Fact	P'	P"
1	6.580	20%	6.596	1.7	0.9454	0.880	0.0165
2	6.519	20%	6.553	3.4	0.8863	0.881	0.0165
3	7.037	20%	7.096	5.9	0.8314	0.873	0.0163
4	7.294	20%	7.380	8.6	0.7757	0.869	0.0162
5	7.306	20%	7.416	11.0	0.7224	0.868	0.0162

1. Check the Brotherton-Ratcliffe & Iben approximation for year 5.
2. Calculate the NPV of the spread (the up-front benefit of in-arrears)
3. Calculate the approximate spread per year, if the 5 year swap rate is 6.656%ab.

Problem 3 - Answers

Yr of pmt	Adj Fwd	Sprd	Disc Fact	NPV (sprd)	B-R&I	NPV(sprd)
1	6.596	1.7	0.9454	1.59	1.6	1.54
2	6.553	3.4	0.8863	2.98	3.2	2.83
3	7.096	5.9	0.8314	4.93	5.6	4.62
4	7.380	8.6	0.7757	6.67	7.9	6.15
5	7.416	11.0	0.7224	7.91	9.9	7.19
		bp up-front		24.09		22.32
		bp/yr approx		5.82		5.39
		bp /yr act		5.70		5.28