# FITTING FORWARD RATES
# TO MARKET DATA - draft

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## INTRODUCTION

The forward curve (or yield curve or term structure of interest rates) is the basic building block for valuing and hedging bonds, swaps, futures, and options. In fact, fitting the forward curve to market data forms the foundation for most of modern fixed income capital markets. Nonetheless, debate and elaboration continues on both the theoretical and practical fronts. (See McCulloch 1971, 1975; Shea 1984; Chambers, Carleton, and Waldman 1984; Vasicek and Fong 1982; Coleman, Fisher, Ibbotson 1992, 1993; Adams and van Deventer 1994 for various approaches to fitting the yield curve.) Indeed, among practitioners there is no consensus on the “right” way to fit the curve and many, when candid, would admit that their firm’s implementation is inadequate in one way or another.

This paper has two purposes. First, to outline a general framework or methodology for fitting the forward curve to market data. The general framework is a practical, market-tested approach that has been developed and refined over the years and has been used at various institutions for pricing, trading, and hedging fixed-income derivatives (swaps) portfolios. Laying out the methodology in a coherent, consistent manner is valuable from a practical perspective (although there is nothing radically new in this approach). A general framework allows one to carry tools and techniques across different products and markets, leveraging investments in technology and intellectual capital. A general framework also pays dividends in terms of writing and maintaining computer code, a substantial benefit given the central role played by mathematical (computer-based) models in pricing and trading fixed income securities. The methodology applies equally to a swap curve, a Treasury or corporate bond curve, or a variety of other curves. Furthermore, this methodology can be used with most of the forward curve functional forms discussed in the literature.

The second purpose of this paper is to report on and compare results from fitting forward curves using three particular functional forms. The first two functional forms, piece-wise constant forward rates and piece-wise linear zero rates, are important because they are commonly used in the markets and because they are simple to implement and to use. Certain relations between the forwards and zero rates for these functional forms are derived and discussed. The third functional form, piece-wise linear forward rates, retains much of the simplicity and ease of use from the first two, while solving a problem (large jumps in the instantaneous forward rates) exhibited by them. Results are reported for US dollar swap curves for October 1994 and June 1997.

The organization is as follows. First the theory of forward and zero curves is briefly reviewed, and then the general methodology for fitting yield curves is introduced and discussed. Second, the uses of fitted curves is discussed. This is important because the use of the curve helps determine an appropriate functional form for the forward curve. This paper focuses on fitting curves for mark-to-market of swap or bond type instruments, but the general fitting methodology applies to a wider class of problems. Third, three specific functional forms for the forward curve are introduced. The first two (piece-wise constant forwards and piece-wise linear zeros) are commonly used in the market. The last (piece-wise linear forwards) is not as common but solves some problems associated with the first two.

## THEORETICAL BACKGROUND

The term structure of interest rates may be stated in several forms; the choice is largely a matter of convenience. The more popular forms are:

- The forward rate. This is used here.
- The zero or spot rate (pure discount rate or yield on zeros). Theoreticians and textbooks commonly discuss this function.
- The discount or present value function (the price of a zero).
- The yield to maturity of par bonds paying periodic coupons. Bond traders are interested in this form.

Mathematically these different forms are equivalent since one can translate between them. With any of these representations of the term structure, however, there is a fundamental assumption: There is one underlying forward curve that prices all instruments of a particular class, possibly with some error. In other words, the present value (price plus accrued) of any fixed cash-flow instrument, say bond $k$, can be expressed as:

(1) $$PV_k = \Sigma_j \, CF_j \, df[t_j ; \{F\}] + \varepsilon_k$$

where

$CF_j$ = cash flow at maturity $t_j$
$df[t_j; \{F\}]$ = discount factor for maturity $t_j$ (today = 0), calculated from the curve $\{F\}$
$\{F\}$ = forward curve which does not vary from one instrument to another.
$\varepsilon_k$ = observation (or other) error, may be assumed zero.

The important thing at the moment is not the specific form of $\{F\}$, but rather that it is the same across all bonds. (The choice of whether to use a forward or spot or discount curve, while often significant from a practical perspective, is irrelevant for the moment.) For later reference, the relations between the forward curve, the discount curve, and the spot or zero curve are:

(2a) $$df(t) = \exp[-\textstyle\int_0^t f(u)\, du]$$

(2b) $$y(t) = [\textstyle\int_0^t f(u)\, du] / t$$

where

$f(u)$ = instantaneous continuously-compounded forward rate at u.
$df(t)$ = discount factor for a period $t$ in the future (today = 0)
$y(t)$ = continuously-compounded zero yield from today to t
$t$ = maturity from today, so today = 0. If $t$ is measured in years, then $f(u)$ and $y(t)$ are both continuously-compounded annual rates.

## GENERAL APPROACH TO FITTING THE YIELD CURVE

The general approach to fit forward curves is split into three components:

1. Choosing a parametric functional form for the forward curve, zero curve, or discount curve, which by equation (2) defines a parametric form for the discount factor function.
2. Choosing market data (inputs) and appropriately describing the instruments.

3. Defining and implementing an appropriate objective function and fitting methodology that specifies how to use the market data to calculate numerical values for the parameters of the forward curve.

To make this general approach concrete I will consider the example of fitting a par swap curve, using two year, three year, and five year swaps as inputs.

DISCOUNT FACTOR FUNCTION

The foundation of everything is the Discount Factor function, df[t ;{F}] from above. This function takes two arguments: 1) a date or maturity from today (t), and 2) a set of curve parameters {F} which Specify both the parameters and the functional form of the forward curve. The function df[t;{F}] returns the discount factor for time t in the future. This function fully encapsulates the forward curve.

For the example of fitting the par swap curve I will assume that the forward rates are piece-wise constant:

$$(3a) \qquad f(t;f_1,f_2,f_3) = \begin{cases} f_1 & 0 \le t<2 \\ f_2 & 2\le t<3 \\ f_3 & 3\le t \end{cases}$$

The discount factor for a date t years in the future would be:

$$(3b) \qquad df[t; f_1,f_2,f_3] = \begin{cases} \exp[-f_1 * t] & 0\le t<2 \\ \exp[-2*f_1 - f_2*(t-2)] & 2\le t<3 \\ \exp[-2*f_1 - 1*f_2 - f_2*(t-3)] & 3\le t \end{cases}$$

This is a function of the parameters $(f_1,f_2,f_3)$. These three parameters, together with the specific functional form in (3b), would be enough to fully describe the discount factor function. In terms of practical implementation, $df(t;f_1,f_2,f_3)$ would be a subroutine taking as arguments the maturity t, the parameters $(f_1,f_2,f_3)$, and possibly an identifier specifying the specific functional form (3b).

The assumption about the functional form of the forward curve is not trivial. Because the term structure of interest rates is never observed at every maturity t it is not possible to infer the forward curve directly from the data, to simply “draw” the function df(.) from the data. In a sense the functional form assumption fills in where data are missing. At one extreme one observes market instruments at sparse points with little information on market levels between these points. (For example there are on-the-run Treasuries at five and 10 years, but not between.) An assumption must be made about the shape of the forward curve between these points and this assumption will to a certain extent determine the forward rates between market observations. In this case the forward curve fitting serves as an interpolator between sparse data. At the other extreme one may observe many instruments but each with some error. An assumption about the general shape of the forward curve would serve to smooth out the errors, with each instrument contributing something to the determination of the exact shape. Here the forward curve fitting serves as a smoothing process across noisy data.

## MARKET DATA

The market data consists of two components. First, the static description of an instrument, such as instrument type, maturity, coupon frequency, etc. Second, the specific market price or rate.

The static description of the instrument is used to build a PV function which returns the price or market rate when a forward curve is provided. For most fixed cash-flow instruments this will simply be a sum of discounted cash flows, with the instrument details determining the size and dates of the flows. In terms of a formula the PV will generally be represented as in equation (1):

$$PV_k = \Sigma_j \, CF_j \, df[t_j \, ; \{F\}] \quad . \tag{1}$$

For implementation this must be translated into a computer program with a detailed specification of the cash flow sizes and dates. The instrument subroutine encapsulates institutional details such as coupon frequency, maturity, relevant holidays, day-count basis of the fixed and floating side of a swap, etc.

Some instruments (for example Euro-dollar futures) do not fit exactly into the paradigm of a sum of fixed cash flows. A more general specification, which includes equation (1) would be

$$\text{Price/Rate} = \text{function[ instrument details, df[.; \{F\}] ]} + \varepsilon_k \quad . \tag{1a}$$

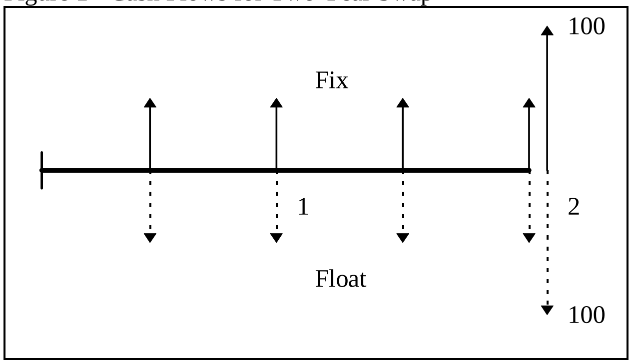
The important aspect is that the forward curve enters only through the discount factor function. Structured in this way, the functional representation of the forward curve (encapsulated in the function df(.)) and the institutional details of the instrument are separated. This strict separation allows considerable generality in curve fitting. It allows the same methodology and code to be applied to a wide variety of instruments, markets, and forward curve functional forms. The assumed functional form of the forward curve can be changed without re-coding instruments, and instruments can be changed without changing the functional representation of the curve.

In the example of fitting a par swap curve, the market data consists of the description and market rates for the par swaps. The market rates are:

| **Type** | **Matur** | **Freq** | **Rate** |
|---|---|---|---|
| Swap | 2 yrs | Semi | 6.36 |
| Swap | 3 yrs | Semi | 6.50 |
| Swap | 5 yrs | Semi | 6.66 |

To develop the PV function, consider the cash flows for the two year swap shown in figure 1. A final exchange of principal is assumed. (Most swap contract do not incorporate a final exchange of principal, but since only net cash flows are paid on swaps this results in the same net cash flows and thus the same PV.) The PV of the fixed side will be the PV of the fixed cash flows, discounted with the appropriate discount factors. The floating side is a libor floating rate bond discounted at libor, which must have PV of par (100).

Figure 1 - Cash Flows for Two Year Swap




Assuming for now that the fixed side payments are exactly half the quoted rate and that payments are exact half-years in the future, the PV function is

$$\text{(1b)} \quad \mathrm{PV}(\mathrm{rate}, \mathrm{yrs}; f_1, f_2, f_3) = \sum_{i=1}^{2*\mathrm{yrs}} \mathrm{df}(i/2; f_1, f_2, f_3) \cdot \mathrm{rate}/2 + 100 \cdot \mathrm{df}(\mathrm{yrs}; f_1, f_2, f_3) - 100$$

We have now completely specified the market data by the combination of the description of the swap instruments (the PV function 1b) and the market data (the swap rates).

FITTING THE CURVE

Once we have defined the discount factor function (the functional form of the curve) on the one hand, and the instruments on the other, the curve must be fit to the market. This requires two things

1. An objective function which defines what one means by a good fit to the market
2. A method of calculating the curve parameters which satisfy the objective function.

For generality we can write this as an operation G which takes us from market instruments to forward curve parameters:

(4) G: {prices/rates; instrument specifications} $\rightarrow$ (p)

The specifics of the operation G will differ depending on the curve functional form and market instruments. The three I have used are

- Exact fit to market data as the objective function and successive single-dimension root-finding as the method of calculating curve parameters. This is appropriate for simple curve forms such as piece-wise constant forwards or piece-wise linear zeros where the number of parameters is the same as the number of instruments.

- Sum of squared differences between actual and predicted prices as the objective function and least-squares minimization as the calculation method. This is appropriate where the number of instruments is larger than the number of parameters, for example fitting a relatively smooth forward curve to noisy US Treasury bonds and bills (as in Coleman, Fisher, Ibbotson, 1992, 1993).

- Sum of squared differences between actual and predicted prices plus squared jumps in forwards plus squared differences in slopes as the objective function and least-squares minimization as the calculation method. This is used for a curve with piece-wise linear (smoothed) forwards, as discussed more fully below.

Continuing with the example of fitting the swap curve, we can use the first of these three methods. To summarize:

- The forward curve is piece-wise constant, with a break at each instrument maturity.
- The two year, three year, and five year par swaps are the relevant set of market instruments.
- The objective function is to choose the parameters so that the PVs of the swaps are all zero.

The objective function or fitting method is to solve the following three equations:

$f^*_1$ s.t. $PV(6.36, 2; f_1) = 0$

$f^*_2$ s.t. $PV(6.50, 3; f^*_1, f_2) = 0$

$f^*_3$ s.t. $PV(6.66, 5; f^*_1, f^*_2, f_3) = 0$

The functional form of the forward curve means that the PV of the two year swap is a function of only the first forward rate, so the first equation can be solved by a simple root-finding. Once the first forward rate has been determined, the PV of the three year swap is a function of only the second forward rate; the second equation can also be solved by a simple root-finding. Given the first and second forward rates, the third can similarly be solved by a simple root-finding. Expressing the forward rates as semi-annually compounded annual rates the solutions are:

$f^*_1$ = 6.360%sab
$f^*_2$ = 6.809%sab
$f^*_3$ = 6.944%sab

DISCUSSION OF GENERAL APPROACH

The modularity of the general approach (separating the forward curve, market data, and fitting method) simplifies modifications or changes to the curve and inputs. In the example above simple bootstrapping would give the same results but could not be altered easily to accommodate different instruments, different curves, or different markets.

- Properly accommodating instrument details is simplified. Cash flow details are encapsulated in the PV function but do not enter either the discount factor function or the fitting method. In the example above, actual US dollar swap payments are commonly on a 30/360 day basis. Incorporating this requires modification of the PV function only, leaving the discount factor and curve fitting method unchanged. (Correctly incorporating day counts, holidays, etc. changes the forward rates to 6.359%sab, 6.789%sab, 6.942%sab for a curve with settle date 2-jul-97.)

- Different or odd instruments can be more easily added. Incorporating Eurodollar deposits or futures requires writing a PV function and simple modification to the assumed forward curve (adding forward rates for the deposit or futures maturities). It does not require changes to the fitting method. (Futures dates generally do not match with swap or deposit dates which can cause problems with simple bootstrapping.)

- Fitting a curve for a different swap market is straightforward. For example, to go from the US where swaps are semi-annual to Germany where swaps are annual requires only a modification to the PV function. The discount factor function and the fitting method are unchanged.

- Moving to a completely different market is simplified. To fit a US Treasury curve the same discount factor function and fitting method could be used, with only the PV function re-written to incorporate the cash flow amounts and dates of the Treasury instruments. This is straightforward even though the coupons for different Treasuries do not all fall on the same dates, a problem with using simple bootstrapping.

- Changing the forward curve functional form is simplified. To use a curve which assumes piece-wise linear zeros instead of piece-wise constant forwards in the example above requires a change to the discount factor function only, with no change to the PV function or the fitting method. (Fitting a piece-wise linear zero curve implies zero rates of 6.359%sab, 6.502%sab, 6.680%sab at two, three, and five years.)

## DIFFERENT USES OF AND CRITERIA FOR FITTED CURVES

Fitted curves are primarily used for one of two purposes: First, as an interpolator between sparse data for mark-to-market, risk measurement, and hedging a portfolio; and second as a smoothing process across noisy data for rich - cheap analysis of a market or portfolio.

### INTERPOLATOR FOR MARK-TO-MARKET

When a curve is used for mark-to-market and risk analysis of a portfolio it is primarily an interpolator, and the following criteria come to the fore:

- There are relatively few inputs, but each input is liquid, is measured without error, and must be fit exactly. (In building the US swap curve out to 30 years there might be three deposits, 16 futures, and eight par swaps for a total of 27 input instruments. No auditor would be happy with such a curve that did not price a Euro-dollar futures or a seven year swap back to the observed market price.)

- Speed and simplicity of computation is important. The curve is created and recreated frequently during the day as the market changes.

- Strong localization of effect is important, in the sense that if an input instrument changes it has an impact only on nearby forward rates. (For example, if a swap curve is built using 10, 12, and 15 year swap rates and the 12 year market swap rate changes with no change in the 10 or 15 year rates, there should be no appreciable change in forward rates at eight or nine years.) The reason goes back to hedging. An eight year maturity instrument would be hedged with a seven and 10 year swap, and should have no appreciable sensitivity to a 12 year instrument. The curve building methodology should incorporate this localization property.

- The forward rates should be reasonably smooth and should not oscillate too much. Often instruments being marked in the portfolio are forward instruments, and odd behavior of the forward rates translates directly into odd behavior of the mark-to-market.

In summary, for mark-to-market and risk analysis the forward curve is primarily used as an interpolator between liquid or on-the-run instruments: The methodology should produce reasonable interpolation at a reasonable cost.

## SMOOTHING FOR RICH-CHEAP ANALYSIS

When a curve is used for rich-cheap analysis, the criteria change:

- There are generally a large number of input instruments and not all will fit the curve exactly. (For example, fitting a US Treasury curve might involve upwards of 200 bonds.)
- The forward curve represents some unobserved "market" curve, and certain restrictions (such as smoothness of forwards) are imposed for theoretical reasons. The purpose of fitting the curve is to smooth noisy data and identify instruments which appear to be miss-priced relative to the "market". The underlying presumption is that not all instruments fit exactly.
- Weak localization of effect is generally required, but strong localization cannot be imposed. Strong localization (changes in an instrument have no impact beyond the neighboring instruments) is contrary to the objective of fitting a "market" curve which measures the relative value of instruments. (By weak localization I mean that a change in a one year bond should have a minimal effect on forwards at 20 years. This was a problem in some early curve-fitting methods. (See Chambers, Carleton, Waldman 1984, and Shea 1984 for a discussion.))
- Speed and simplicity are often less important, since the curve is usually not rebuilt frequently.

In summary, for rich-cheap analysis more a-priori assumptions about the curve are generally imposed, and some minimization technique such a least-squares is used to calculate the parameters of some "smooth" curve.

## FUNCTIONAL FORMS FOR DIFFERENT APPLICATIONS

Some functional form assumptions are more appropriate for mark-to-market while others are more appropriate for rich-cheap analysis. The functional forms discussed below (constant forwards, linear zeros, and linear forwards) are particularly appropriate for mark-to-market: they all provide, to a greater or lesser extent, good interpolation, speed of computation, and localization of effect. (In fact, the constant forwards and linear zeros are probably the most popular among practitioners in the derivatives markets.) On the other hand, cubic splines and linear (smoothed) forwards are more appropriate for rich-cheap analysis since these provide better smoothing with less strong localization.

Note that the specific functional forms I have chosen to focus on differ in one important respect from those often used in fitting the yield curve: Continuity and smoothness of forward rates across the whole curve is not imposed a priori. Many studies of yield curve fitting impose smoothness (see, for example, Vasicek and Fong 1982; Adams and van Deventer, 1994) and smoothness across knot points is often cited as an advantage of cubic splines. Although finite period forward rates should not jump, I know of no argument to rule out a finite number of discontinuities (jumps) in the instantaneous forward rate function. Zeros must be continuous, but since zeros are integrals of forwards this is ensured as long as instantaneous forwards are piece-wise continuous (have only a finite number of jumps). In fact, there are reasonable arguments why instantaneous forwards should be discontinuous at certain points. Take the case of a Federal Reserve Open Market Committee meeting at a known future date, with market expectations of a possible rate hike (a discontinuous change in short-term interest rates). In this case I would think there should be a discontinuity in instantaneous forwards.

**SPECIFIC FORWARD CURVE FUNCTIONAL FORMS**

For the actual fitting of a forward curve, the functional form of $\{F\}$ must be specified with a set of parameters undetermined. Mathematically, we want to write the forward curve $\{F\}$ from above as a specific functional form, dependent on a set of parameters $p$: $\{F(p)\}$. The data (prices of traded instruments) would then be used to determine the parameters.

I focus on three functional forms for forward rates (the last two are variants of each other):

1. Piece-wise constant forwards - the instantaneous (continuously-compounded) forward rate between each of a set of chosen knot or break points (usually the maturity dates of chosen instruments) is constant.

2. Piece-wise linear zeros - the continuously-compounded zero rate between each of a set of chosen knot or break points is linear, and the zeros are continuous (but not smooth) across break points. This leads to piece-wise linear forward rates which are discontinuous (with sometimes large jumps) at break points.

3a. Piece-wise linear (twisted) forwards - the instantaneous forward rate between each of a set of chosen break points is linear, with the slope chosen to equal the slope between the average forward rates over the two adjacent periods.

3b. Piece-wise linear (smoothed) forwards - as for (3a) the instantaneous forward rate between each of a set of chosen break points is linear. The objective function and minimization routine used to choose the parameters (see below) is chosen to provide a smoothed forward curve; i.e. to minimize (but not eliminate) the jumps at break points.

These functional forms are appropriate for mark-to-market applications. The first two are particularly simple to implement and use, and are popular with traders in the derivatives markets. Furthermore, they imply simple and intuitive curve risk measures. The third specification is important because it solves a problem (unreasonable jumps in the forward rates) exhibited by the first two and has worked well in the actual pricing, trading, and hedging of a derivatives portfolio. Note, however, that although these three forward curve functional forms are important, the general curve fitting framework discussed here is not restricted to these.

I do not discuss cubic splines as a functional form because it is often a poor choice for building a mark-to-market curve. The smoothness imposed (the forward curve is assumed $C^1$ or $C^2$) can lead to severe non-localization and is not necessary. Cubic splines may be appropriate, however, for rich-cheap analysis where smoothing is relatively more important.

PIECE-WISE CONSTANT FORWARDS

For the piece-wise constant curve (shown in figure 2), break points ($b_1$ , …, $b_n$) measured as time from today are chosen a priori. The parameters would be the constant instantaneous forward rates ($f_1$ , …, $f_n$) between the break points. We can write the curve as $\{F(f_i)\}$.

The discount factor for a period t in the future would be as given in (2a) above:

$$df(t) = \exp[-\int_0^t f(u)\,du]$$

where

f(u) = instantaneous continuously-compounded forward rate at u.

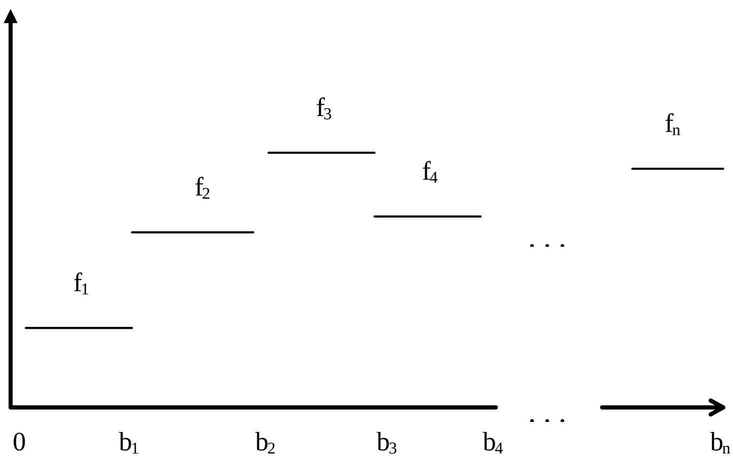
This simplifies to:

$$df[t;\{F(f_i)\}] = \exp[-\Sigma^k_{j=1} f_j *(b_j - b_{j-1}) - f_{k+1}(t-b_k)] \quad (5)$$

where

k is the index such that $b_k < t$.

Figure 2 - Piece-Wise Constant Forward Interest Rate Curve




To apply this to actual data, we might use non-linear least squares to calculate the ($f_i$ ) by minimizing the difference between the actual and predicted prices. The predicted price of instrument k would be:

$$PV_k(f_i) = \Sigma_j CF^k_j \, df[t^k_j; \{F(f_i)\}]$$

where

$CF^k_j$ = cash flow at maturity $t_j$ for bond k

$df[t^k_j; \{F\}]$ = discount factor for maturity $t^k_j$, calculated from the (forward) curve $\{F(f_i)\}$

$\{F(f_i)\}$ = forward curve which does not vary from one instrument to another and depends on the parameters ($f_i$)

The actual price (plus accrued interest) would be $PV_k$ . The minimization would be:

$$\text{Min}_{fi} \ \Sigma_k \, [PV_k(f_i) - PV_k \,]^2 \, w_k$$

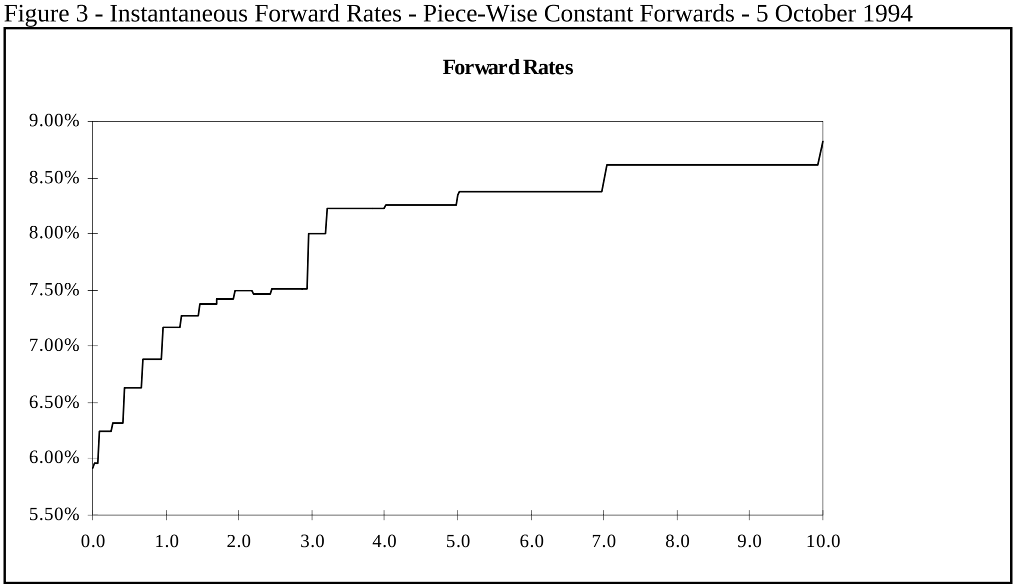
where

$(f_i)$ = parameters to be estimated
k = index running over traded instruments
$PV_k(f_i)$ = PV of instruments k as a function of the forward curve parameters $(f_i)$
$PV_k$ = market price of instrument k
$w_k$ = (possible) weighting function applied to instrument k

If the breaks were chosen so that there was a break at the maturity of each instrument, then the minimization problem would reduce to the problem of finding the zeros of a sequence of single dimensional functions, since there would be as many parameters as instruments (i.e. as many rates $f_i$ as prices $PV_k$). This gives a "point-to-point" or "bootstrap" exact fit to the observed prices.

The following figure shows the fitted forward curve (instantaneous forward rates) for the USD swap market for 5 October 1994 using piece-wise constant forward rates (see below for details of the data and results). The forward rates are well behaved but show jumps at the break points. Intuitively it would make sense to simply "twist" the forwards to help smooth the forward rates across jumps.

Figure 3 - Instantaneous Forward Rates - Piece-Wise Constant Forwards - 5 October 1994




PIECE-WISE LINEAR ZEROS

Break points are chosen a priori, as for the piece-wise constant forward curve. The zero rate from t=0 to the first break is assumed constant. Thereafter the continuously-compounded zero rates are assumed linear between break points. In other words zero rates are linear and continuous but not smooth across break points. As shown below, this assumption leads to instantaneous forward rates which are piece-wise linear and discontinuous across break points.

The restriction that zero rates are linear between break points means that the discount factor function can be written as:

(6) $$df(t;\{F(p_i)\}) = \exp[-y(b_k + x)*t]$$

(7) $$y(b_k + x) = y_k\,[1 - x/(b_{k+1}-b_k)] + y_{k+1}x/(b_{k+1}-b_k)\ .$$

where

k is the index such that $b_k < t$.
$x = b_{k+1} - t$ ; i.e. the distance from the last break to maturity
$y_k$ = zero at break before maturity
$y_{k+1}$ = zero at break after maturity .

Equation (7) expresses the restriction that zeros between breaks are linear. A convenient way of representing the $y_j$ in terms of forward rates is as

(8) $$y_j = [\Sigma^j_{i=1} f_i *(b_i - b_{i-1})] / b_j \quad \text{for } j>1 \qquad \text{and } y_o, y_1 = f_1.$$

where $f_i$ is the average forward rate over the period $[b_{i-1},b_i]$ (also the forward rate at the mid-point of the period). The restriction that zeros are continuous and linear between breaks implies that the forwards are linear but discontinuous. Specifically, the forward rate between breaks i and i+1 is given by:

(9a) $$f(b_i+x) = f_{i+1} + 2*(f_{i+1} - y_i)*[x+(b_i - b_{i+1})/2]/b_{i+1}$$

or

(9b) $$f(b_i+x) = [y_i(b_{i+1}-2b_i) + y_{i+1}b_i + 2x(y_{i+1}-y_i)]/(b_{i+1}-b_i)$$

where

$y_i$, $y_{i+1}$ = zero rate at breaks i, i+1
$b_i$, $b_{i+1}$ = time to breaks i, i+1
x = time from break i; i.e. $0 \le x \le b_{i+1}-b_i$
$f_{i+1}$ = average forward rate over the period (forward rate at the middle)

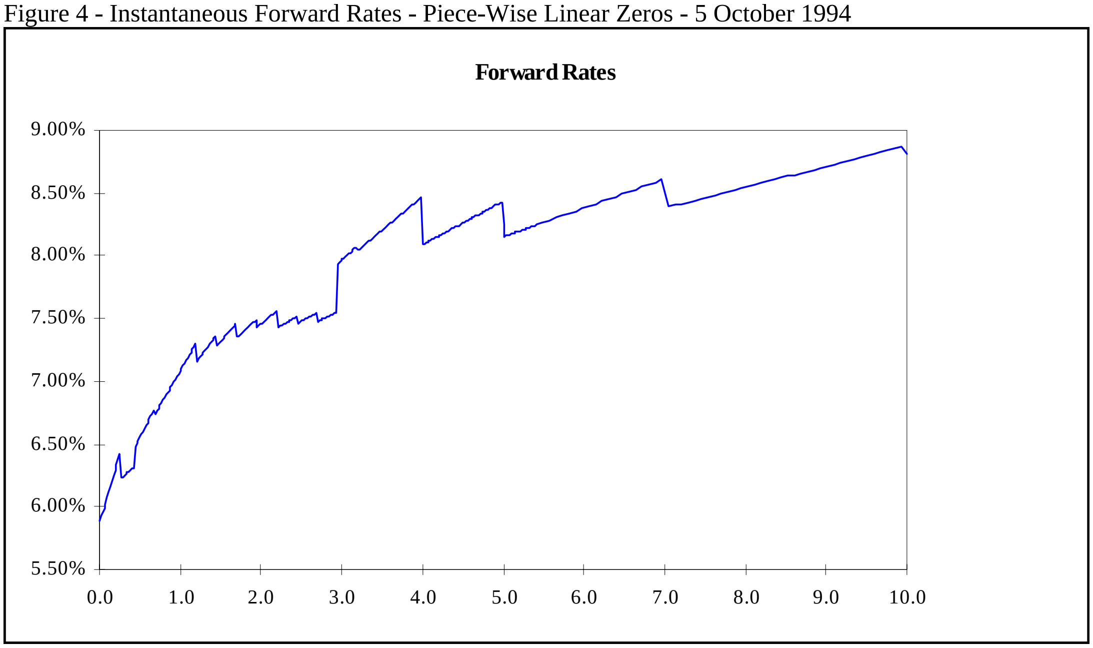
Equation (9a) implies that the instantaneous forward rates are linear but with the restriction that the slope of the forward curve is equal to

$$2*(f_{i+1}-y_i)/b_{i+1}$$

i.e. twice the difference between the average forward rate and the zero rate at the beginning divided by the time to the end of the period. There is no intuition I am aware of for this expression; it is simply the restriction required to ensure that zero rates are linear between break (knot) points. As will be seen later, this restriction on the slope of the forward curve is often unreasonable. The derivations of equations (9a) and (9b) are given in the appendix.

The following figure shows the fitted forward curve (instantaneous forward rates) for the USD swap curve for 5 October 1994 using linear zeros (see appendix for details of the data and results). The forward rates show substantial jumps at the break points. Examination of figure 4 and comparison with figure 3 shows that in many cases the forward rates rise so much during a period that they must jump down to the beginning of the next period: The forwards are “twisted up” too much during many periods. This is a common result with piece-wise linear zeros and is not limited to the particular date chosen.

Figure 4 - Instantaneous Forward Rates - Piece-Wise Linear Zeros - 5 October 1994




PIECE-WISE LINEAR FORWARDS (TWISTED AND SMOOTHED)

Break points ($b_1$ , …, $b_n$) are chosen a priori, usually equal to the maturities of the market instruments. Each forward rate period (except the last) has two parameters: the forward rate at the middle of the period (or the average for the period) ($f_i$) and a slope ($s_i$). The last forward rate period has only a constant forward rate, $f_n$. In other words the forward rate is

(18) $$f(u) = f_i + s_i * [ u - (b_i + b_{i-1})/2] \qquad b_{i-1} < u \le b_i \ .$$

with $s_n \equiv 0$ .

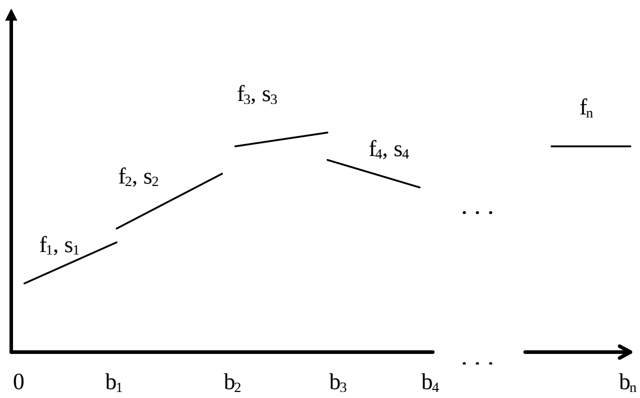
The general expression for the discount factor for a period t in the future is given in (2a) above, and here this reduces to:

(19) $\quad df[t;\{F(f_i)\}] = \exp[-\Sigma^{k}_{j=1} f_j *(b_j - b_{j-1}) - f_{k+1}(t-b_k) - s_{k+1}(b_{k+1}b_k - b_k t - b_{k+1}t + t^2)/2]$

where

k is the index such that $b_k < t$.

Figure 5 - Piece-Wise Linear (Smoothed) Forward Interest Rate Curve




To apply this to actual data where there is one market instrument per forward rate period further restrictions must be imposed. (There are more parameters than data points.) The restrictions I have used are of two sorts. The first (what I call "twisted" forwards) is to set the slope equal to the weighted average of the slope between the two adjacent (average) forward rates, using the lengths of the forward periods as weights. For k>1 (and remembering that $s_n \equiv 0$ and defining $b_0=0$) the slope $s_k$ will be:

$$s_k = [s^- * \text{years}(f_{k-1} \rightarrow f_k) + s^+ * \text{years}(f_k \rightarrow f_{k+1})] / \text{years}(f_{k-1} \rightarrow f_{k+1})$$

where

$s^-$ = average change from $f_{k-1}$ to $f_k$
$s^+$ = average change from $f_k$ to $f_{k+1}$

$$s^- = (f_k - f_{k-1}) / [(b_k - b_{k-2})/2]$$

$$s^+ = (f_{k+1} - f_k) / [(b_{k+1} - b_{k-1})/2]$$

$$\text{years}(f_{k-1} \rightarrow f_k) = (b_k - b_{k-2}) / 2$$

$$\text{years}(f_{k-1} \rightarrow f_{k+1}) = (b_k - b_{k-2}) / 2 + (b_{k+1} - b_{k-1}) / 2$$

$$s_k = [(f_k - f_{k-1}) + (f_{k+1} - f_k)] / [(b_k - b_{k-2}) / 2 + (b_{k+1} - b_{k-1}) / 2]$$

$$s_k = [f_{k+1} - f_{k-1}] / [(b_k - b_{k-2}) / 2 + (b_{k+1} - b_{k-1}) / 2]$$

For k=1,

$$s_1 = [f_2 - f_1] / [(b_2 - b_0) / 2]$$

This makes the number of parameters ($f_1, \ldots, f_n$) equal to the number of market instruments. The parameters can be calculated either using non-linear least squares (on the sum of squared differences between actual and predicted prices) or by multiple passes of sequential single-dimensional root-finding.

The second set of restrictions I have used is to assess a penalty against both jumps in the instantaneous forward rates and changes in the slopes. In other words, the objective function is a combination of

- Sum of squared differences between actual and predicted market prices
- Sum of squared jumps in the instantaneous forward rates across breaks
- Sum of squared differences in the slopes from period to period.

The latter two parts of the objective function impose the restriction that forward rates should tend to be smooth (by minimizing jumps and curvature or zig-zagging) while the first part imposes the restriction that the curve fit the market data. Written mathematically the overall objective function is:

$$\mathrm{Min}_{(fi,si)}\ \Sigma_k\ [PV_k(f_i,s_i) - PV_k\ ]^2\ w_k\ +\ w_{jump}\ \Sigma^n{}_{j=2}\ [(f_j - s_j(b_j - b_{j-1})/2) - (f_{j-1} + s_{j-1}(b_{j-1} - b_{j-2})/2)]^2$$

$$+\ w_{slope}\ \Sigma^n{}_{j=2}\ [s_j - s_{j-1}]^2$$

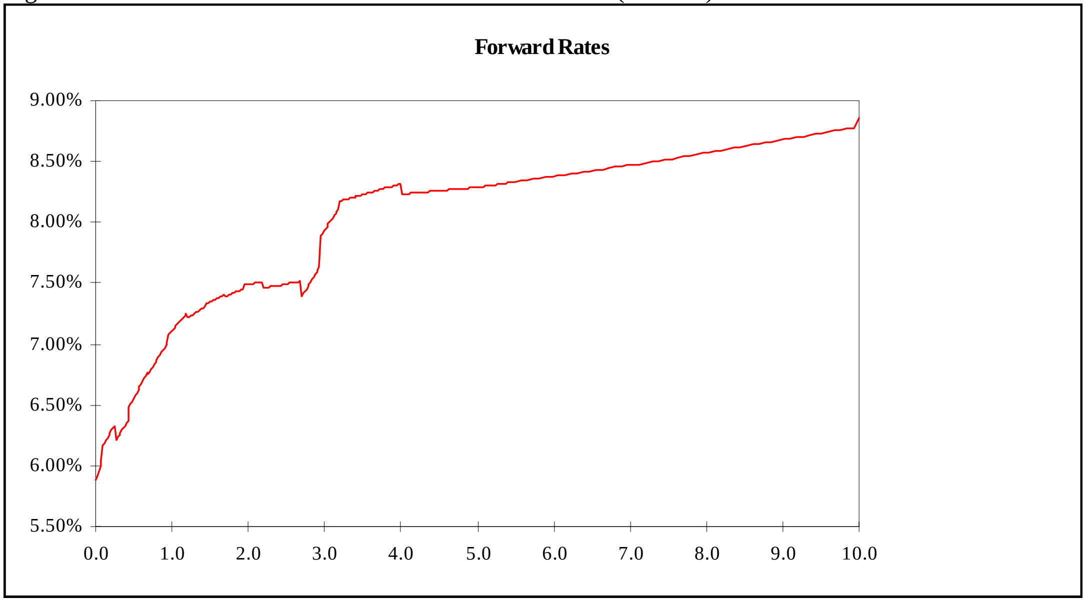
where

$(f_i, s_i)$ = parameters to be estimated, with $s_n \equiv 0$
k = index running over traded instruments
$PV_k(f_i)$ = PV of instrument k as a function of the forward curve parameters $(f_i, s_i)$
$PV_k$ = market price of instrument k
$w_k$ = (possible) weighting function applied to instrument k
$w_{jump}$ = relative weighting applied to sum of jump differences
$w_{slope}$ = relative weighting applied to sum of slope differences

Non-linear least squares is used to calculate the parameters.

The following figure shows the fitted forward curve (instantaneous forward rates) for the USD swap market for 5 October 1994 using linear (twisted) forwards (see appendix for details of the data and results). Here the forward rates are substantially smoothed; they show neither the flatness and jumps of the piece-wise constant curve form nor the “over-twisting” and large jumps exhibited by the piece-wise linear zero form. This functional form largely solves the problem of jumps at break points, while still maintaining substantial localization of effect. The piece-wise linear (smoothed) forwards perform a slightly better job at smoothing the forwards, but this is bought at the cost of slower speed of computation.

Figure 6 - Instantaneous Forward Rates - Piece-Wise Linear (Twisted) Forwards - 5 October 1994




**RESULTS FOR PIECE-WISE CONSTANT AND LINEAR FORWARD CURVES**

This section reports results for fitting market data from the US swap market for two dates, 5 October 1994 and 30 June 1997. Both dates show an upward sloping yield curve but with different degrees of slope: For October 1994 the 5-30 spread was 76bp while for June 1997 it was only 41bp. In addition, the curve for October 1994 is monotonic out to 30 years while for June 1997 there is a "hump" at the long end with 30 and 40 year par swap rates lower than 25 year par rates.

INPUT MARKET DATA

Table 1 shows the market data for these two dates. The choice of instruments is determined by the intended use of the curve and the liquidity of available instruments. Here the curve is intended for mark-to-market and hedging of a swap and options portfolio. This implies that liquid instruments with good market quotes should be used. In the US market this means some combination of Eurodollar deposits (libor), Eurodollar futures, FRAs, and par swaps. For October 1994 the choice was deposits out to three months, futures from 2.25 months to 38.4 months, and par swaps from four years out to 30 years. For June 1997 the choice was deposits only for the first week or two, two monthly futures, quarterly futures from 2.6 months to 50.7 months, and par swaps from five years out to 40 years.

Table 1 - US Swap Market Data For October 1994 and June 1997

| | 5-Oct-1994 Bond | Spread Vol | All-in | 30-Jun-97 Bond | Spread Vol | All-in |
|---|---|---|---|---|---|---|
| O/N | | | 5.81 | | | 5.5 |
| 1wk libor | | | | | | 5.6875 |
| 1mth libor | | | 5.9 | | | |
| 3mth libor | | | 6.12 | | | |
| Mthly Fut1 | | | | | 9.00% | 94.31 |
| Mthly Fut2 | | | | | 9.00% | 94.265 |
| Qtly Fut1 | | 15% | 93.74 | | 9.00% | 94.12 |
| Qtly Fut2 | | 15% | 93.41 | | 10.47% | 93.9 |
| Qtly Fut3 | | 15% | 93.14 | | 11.42% | 93.805 |
| Qtly Fut4 | | 15% | 92.86 | | 13.12% | 93.68 |
| Qtly Fut5 | | 15% | 92.76 | | 14.65% | 93.57 |
| Qtly Fut6 | | 15% | 92.65 | | 16.11% | 93.46 |
| Qtly Fut7 | | 15% | 92.59 | | 17.51% | 93.44 |
| Qtly Fut8 | | 15% | 92.51 | | 16.46% | 93.4 |
| Qtly Fut9 | | 15% | 92.53 | | 16.85% | 93.35 |
| Qtly Fut10 | | 15% | 92.49 | | 17.50% | 93.27 |
| Qtly Fut11 | | 15% | 92.47 | | 18.17% | 93.27 |
| Qtly Fut12 | | 15% | 91.97 | | 16.51% | 93.23 |
| Qtly Fut13 | | | | | 16.57% | 93.19 |
| Qtly Fut14 | | | | | 16.76% | 93.12 |
| Qtly Fut15 | | | | | 16.93% | 93.12 |
| Qtly Fut16 | | | | | 16.26% | 93.08 |
| 4yr swap | 7.16 | 33 | 7.49 | | | |
| 5yr swap | 7.36 | 29 | 7.65 | 6.389 | 27.5 | 6.664 |
| 7yr swap | 7.492 | 37 | 7.862 | 6.437 | 33.5 | 6.772 |
| 10yr swap | 7.69 | 38 | 8.07 | 6.509 | 37.5 | 6.884 |
| 12yr swap | 7.69 | 49 | 8.18 | 6.5376 | 41.25 | 6.9501 |
| 15yr swap | 7.69 | 58 | 8.27 | 6.5805 | 44 | 7.0205 |
| 20yr swap | 7.69 | 68 | 8.37 | 6.652 | 44.25 | 7.0945 |
| 25yr swap | | | | 6.7235 | 40.25 | 7.126 |
| 30yr swap | 7.89 | 52 | 8.41 | 6.795 | 32 | 7.115 |
| 35yr swap | | | | 6.795 | 31 | 7.105 |
| 40yr swap | | | | 6.795 | 31 | 7.105 |

The factors determining the specific instrument choices included:

- Short end and deposits - Eurodollar deposits (libor) are important because most swap floating rate payments are set off libor. On the other hand deposits, particularly beyond three months, are not very liquid so do not provide as good hedges or as good market prices as some other instruments (e.g. futures or FRAs). For October 1994 we chose to use deposits only out to the first quarterly futures contract. For June 1997 the monthly futures were liquid enough that we decided to use monthly futures instead of one month and three month deposit rates.

- Middle sector and futures vs. FRAs - For the middle sector of the curve (three months to four or five years) the choice is between futures and FRAs. Either one is suitable. Factors in favor of futures are the high quality and transparency of futures prices (settlement prices are published daily by the exchanges) and the common use of futures to hedge (using futures to build the curve insures they are priced correctly off the curve). A factor against futures is the different convexity characteristics of futures versus FRAs and swaps, which requires a convexity adjustment when futures are used. How far to use futures or FRAs depends partly on the liquidity of the further out contracts and partly on how far futures or FRAs are used for hedging. For October 1994 we used three years of futures while by June 1997 we used four years of futures.

- Long end - For the longer end of the curve the only choice is par swaps. We used the most liquid par swaps as input points, and relied on the curve as an "interpolator" to price less liquid points (such as 17 year par swaps).

The general approach to fitting the curve outlined in this paper simplifies the process of switching input instruments (say from using three years to four years of futures or from using FRAs to futures). The decision of which instruments to use as inputs should be driven by considerations of instrument liquidity, quality and transparency of prices, and use as hedging instruments, not implementation issues.

## INSTRUMENT PV FUNCTIONS

The fitting methodology requires building a valuation function (PV function) for each instrument, given a discount factor function. Eurodollar (libor) deposits and swaps are both fixed cash-flow instruments and so building a PV function is straightforward.[1] For Eurodollar futures building a PV function it is not so straight-forward. The major issue is the difference in convexity between futures and FRAs. Deposits and swaps are both equivalent to FRAs in the sense that one can be constructed from the other - arbitrage insures the relative pricing is consistent. (For example a six month libor is the same as, and must trade at the same price as, a three month deposit plus a 3x6 FRA.) The possibility of direct arbitrage ensures that Eurodollar (libor) deposits, FRAs, and swaps are all valued directly off the same curve.

Eurodollar futures are a different case. The linear pay-off for futures (daily mark-to-market) leads to a futures contract having a different convexity than an FRA (see Burghardt and Hoskins, 1995a, b, c). This implies that a futures contract cannot be valued directly off an FRA/swap curve. One common approach is to apply a "convexity correction" to the futures rate which effectively converts it to an FRA rate. The correction I am using (taken from Doust (1995)) is:

$$R_{fut} = R_{fra} / \{DF_{start}\text{^}[½\sigma^2 t*(t+½)/(t+¼)]\}$$

where

$t$ = time to futures expiry (in years)
$R_{fra}$ = forward (FRA) rate from the curve
$DF_{start}$ = discount rate to the futures start date (expiry date)
$\sigma$ = volatility in decimal, i.e. measured as 0.20.

## FITTED CURVES

[1] Libor deposits are assumed two day settle (except O/N which is zero days) and quoted on an A/360 basis. Swaps are assumed modified following business day convention, fixed side semi-annual 30/360 day basis, floating side quarterly A/360 day basis. These are the standard quoting conventions for USD swaps.

Three forward curve functional forms are used to fit the market data: Piece-wise constant forwards (PWCF), piece-wise linear zeros (PWLZ), and piece-wise linear forwards (PWLF) (both twisted and smoothed). Table 2 shows the estimated parameters. For October 1994 the first column shows the breaks chosen for the forward curve. These breaks are the maturities of the market instruments used to fit the curve. The second column shows the instantaneous (continuously compounded) forward rates calculated assuming constant forward rates between the break points. The third, fourth, and fifth columns show the parameters assuming the zero curve is piece-wise linear and the forward curve is piece-wise linear (twisted and smoothed), respectively. (The level parameters are almost the same for all curve forms because in each case they are the average forward rate over the period or the forward rate at the middle of the period.) For the PWLF curve the slope parameters are shown at the bottom of the table. The piece-wise forward smoothed curve was fitted using relative weights of 100 for the sum of squared price differences, and 1.0 and 1.0 for the sum of squared jump and slope differences. Note that the slope parameters for the twisted and smoothed forward curves are almost the same except for the slope for the first few periods.

All the forward rate functional forms fit the input data, so that all the input instruments are priced back to the market. There is no difference between the curves on this criterion. There are substantive differences between the fitted curves, however, and these become obvious when one graphs the instantaneous forward rates. Figures 7 through 12 show the instantaneous forward rates and zero rates implied by the three types of curves for the two dates. (For figures 7 through 10 only the twisted version of the linear forwards are shown since the twisted and smoothed versions are so similar.)

The piece-wise linear zero curve has problems with extreme, saw-tooth, jumps in the instantaneous forward rates which do not appear in the piece-wise constant or piece-wise linear forward curves. Given two sets of forward rates which both fit observed market data, there are arguments for using the smoother set of rates even when it is allowed that instantaneous forward rates may be discontinuous. The problems with piece-wise linear zero curves appear severe enough to cast doubt over use of this type of curve for pricing any derivative instrument. Focusing on one section of the October 1994 curve shows the magnitude of the problems. According to the fitted piece-wise linear zero curve, the average forward rate was 8.255% between 3.2 and 4.0 years, 8.262% between 4.0 and 5.0 years, and 8.383% between 5.0 and 7.0. Nonetheless, this curve form implies that the instantaneous forwards rose to 8.46% by the end of year three and jumped down by 37bp to 8.06% at the beginning of year four, only to rise up to 8.43% by the end of year four and jump back down by 27bp to 8.16%. In contrast the piece-wise constant forward curve implies jumps of only 3bp and 8bp across the year boundaries. This reinforces the fact that the large jumps generate by the piece-wise linear zero curve are not implied by the data but are an artifact of restrictions on the slope imposed by the assumption of linear zero rates. (The quarterly forward rates fell by 27bp from the end of year three to the beginning of year four and by 21bp from the end of year four to the beginning of year five. These finite period forward rates are more important because they represent actual traded FRAs or caplets.)

Neither the piece-wise constant forward nor the piece-wise linear forward curves have the problems with extreme jumps exhibited by the piece-wise linear zero curve. The piece-wise constant forward curve has jumps at knot points, but the forward rates do not exhibit the saw-tooth behavior of the piece-wise linear zero curve. At the short end of the curve, where the curve is steep, the piece-wise constant curve has a stair-step pattern. The forward rates do not overshoot as for the piece-wise linear zero curve, but the consistent slope of the curve implies that the jumps at knot points could be made much smaller simply by twisting the forward rates. (In fact, at the short end of the curve, the piece-wise linear curve fits quite well.)

The piece-wise linear forward curve seems to solve both the saw-tooth problem of the linear zero curve and the saw-tooth pattern of the constant forward curve. The twisted version of the linear forward curve sets the slope in a period equal to the average forward curve slope in the region, and this appears to work pretty well. The smoothed version explicitly builds the curve to minimize jumps (while still fitting market prices) but requires more complex optimization routines to calculate the parameters.

## LOCALIZATION AND HEDGING DIFFERENCES

Both the constant forward and the linear zero curve have strong localization, in the sense that a change in an input rate does not alter forward rates beyond the adjacent inputs. The linear forward curve, however, does not exhibit such strong localization - there is a little “bleed” into adjacent forward rate periods. This is a natural result of the slope of the forward rate being dependent on the level of adjacent forward rates. Figure 13 shows the changes in the instantaneous forward rates (for June 1997) resulting from bumping up the input seven year par swap rate by 1bp. The constant forward and linear zero curves show strong localization, with forwards between five and seven years shifting up and forwards between seven and 10 years shifting down and no changes outside the five to 10 year period. This is because the forward rates prior to five years do not depend at all on the forwards after five years. For the piece-wise linear forward curve there is some non-localization with forwards between four and five years and 10 and 12 years shifting. Note, however, that it is only the slopes of the forwards that change and the shift is about six times smaller than the shift in forwards between five and 10 years.

In response to the shift in the seven year input the forward curve flattens as the forwards in the five to seven year period shift up and the forwards in the seven to 10 year period shift down. (The difference between the average forward over the five to seven year period versus the seven to 10 year period is 14bp before the shift.) This can be seen in figure 13 with the difference being positive before year five and negative after year five. For the linear forward curve the slopes flatten, both getting smaller (closer to zero). This means that the instantaneous forwards at 5.1 years shift up by more than the forwards at 6.9 years, while the forwards at 7.1 years shift down by less than the forwards at 9.9 years, as can be seen in figure 13. The situation is different for the linear zero curve. The slope of the forward curve between five and seven years actually increases, although the forward curve itself has flattened. This is because for the linear zero curve a shift up in the average forward rate of $h$ implies a shift up in the slope of $2*h/b_{i+1}$ (see equation 9a).

The changes in the shapes of the forward curves imply differences in hedge ratios for the different curves. The easiest way to see the differences is to focus on a "barbell": hedging an instrument whose maturity falls between two liquid instruments. An example would be hedging a six year swap with a combination of a five and seven or an eight year swap with a combination of seven and 10. The following table shows the hedge ratios for these two barbells under the three curve types. The hedge is measured in two ways. First, as the change in the six year par swap rate when the five or seven swap rates rise by 1bp (0.42bp and 0.58bp for the constant forwards curve), and second as the change in the PV of a par swap ($1mm notional) when the five or seven swap rates rise by 1bp ($205 and $283 for the constant forwards curve).

Hedge Ratios For Barbell Hedge, Various Curves, June 1997

| | Constant Forwards | | Linear Zeros | | Liner Forwards | |
|---|---|---|---|---|---|---|
| | Swap rate | Swap PV | Swap rate | Swap PV | Swap rate | Swap PV |
| Sensitivity of six to | | | | | | |
| Five | 0.42 | $205 | 0.50 | $244 | 0.54 | $263 |
| Seven | 0.58 | $283 | 0.50 | $244 | 0.64 | $312 |
| Both | 1.00 | $487 | 1.00 | $487 | 1.18 | $575 |
| Sensitivity of eight to | | | | | | |
| Seven | 0.57 | $347 | 0.66 | $402 | 0.65 | $396 |
| Ten | 0.43 | $262 | 0.34 | $207 | 0.57 | $347 |
| Both | 1.00 | $609 | 1.00 | $609 | 1.22 | $743 |

For the linear zero (PWLZ) curve the hedge ratio is linear in the distance to the adjacent hedges. The risk of the six year is equally split between the five and seven year swaps while the eight year is split in the ratio 2/3 to the seven year and 1/3 to the 10 year. The linear hedge ratio is a result of the assumption that the zeros between knot points are linearly interpolated. Linear interpolation of the zero rates is close to linear interpolation of the par rates and linear interpolation of par rates implies linear allocation of risk; linear zeros and linear risk go hand-in-hand.

The results for the linear forwards curve are interesting. The six year swap rate rises by 1.18bp when both the five and seven rise by 1bp. Initially this seems counter-intuitive, but it is a result of the flattening of the forward curve. Figure 14 shows what happens. When the five and seven year swap rates rise, the forward rates between four and five years rise while the forwards between seven and 10 years fall. The slope between five and seven decreases (the curve flattens) which means that rates in the first part of the period (five to six years) rise relative to the rates in the second part of the period (six to seven years). Since the average change over the five to seven year period is enough to generate a 1bp rise in the seven year swap rate and the rates in the five to six year period rise more than the rates in the six to seven year period, a six year par rate must rise by more than 1bp. The end result is that a six year or eight year par rate will rise by more than 1bp when rates around them rise by 1bp.

For the constant forwards curve the risk is allocated in a "sub-linear" manner so that the risk of the six year swap is split in the ratio 0.42 / 0.58 while the risk of the eight year swap is split in the ratio 0.57 / 0.43.

**CONCLUSION**

This paper has discussed a general methodology for yield curve fitting which is applicable across a wide variety of markets and applications. I have discussed the application of this methodology to fitting curves for mark-to-market, and discussed three specific functional forms - piece-wise constant forwards, piece-wise linear zeros, and piece-wise linear forwards - in detail. I have argued that the piece-wise linear functional form has many of the advantages of the simpler forms used in the market while solving some of the draw-backs. Examples of applying the methodology and specific forward curve functional forms have been applied to US dollar swap market data for October 1994 and June 1997.

Table 2 - Forward Curve Parameters for Piece-Wise Constant Forwards (PWCF), Piece-Wise Linear Zeros (PWLZ), Piece-Wise Linear (Twisted and Smoothed) Forwards (PWL(T)F and PWL(S)F)

| 10/5/94 | PWCF | PWLZ | PWL(T)F | PWL(S)F | 6/30/97 | PWCF | PWLZ | PWL(T)F | PWL(S)F |
|---|---|---|---|---|---|---|---|---|---|
| 10/7/94 | 5.890 | 5.890 | 5.890 | 6.032 | 7/2/97 | 5.576 | 5.586 | 5.575 | 5.714 |
| 11/7/94 | 5.967 | 5.967 | 5.967 | 6.066 | 7/7/97 | 5.764 | 5.764 | 5.764 | 5.718 |
| 1/9/95 | 6.249 | 6.249 | 6.249 | 6.188 | 8/18/97 | 5.754 | 5.748 | 5.752 | 5.748 |
| 3/14/95 | 6.317 | 6.272 | 6.295 | 6.332 | 9/22/97 | 5.799 | 5.798 | 5.798 | 5.799 |
| 6/13/95 | 6.628 | 6.628 | 6.627 | 6.627 | 12/17/97 | 5.925 | 5.924 | 5.923 | 5.924 |
| 9/19/95 | 6.892 | 6.883 | 6.883 | 6.888 | 3/17/98 | 6.137 | 6.137 | 6.137 | 6.136 |
| 12/19/95 | 7.171 | 7.170 | 7.170 | 7.169 | 6/18/98 | 6.230 | 6.230 | 6.230 | 6.230 |
| 3/19/96 | 7.265 | 7.264 | 7.264 | 7.266 | 9/17/98 | 6.354 | 6.354 | 6.354 | 6.354 |
| 6/18/96 | 7.371 | 7.370 | 7.370 | 7.370 | 12/16/98 | 6.461 | 6.460 | 6.460 | 6.460 |
| 9/17/96 | 7.425 | 7.424 | 7.424 | 7.425 | 3/16/99 | 6.565 | 6.565 | 6.565 | 6.564 |
| 12/17/96 | 7.499 | 7.498 | 7.498 | 7.498 | 6/17/99 | 6.576 | 6.575 | 6.575 | 6.575 |
| 3/18/97 | 7.471 | 7.471 | 7.471 | 7.472 | 9/16/99 | 6.613 | 6.612 | 6.612 | 6.612 |
| 6/17/97 | 7.504 | 7.504 | 7.504 | 7.504 | 12/15/99 | 6.655 | 6.655 | 6.655 | 6.655 |
| 9/16/97 | 7.515 | 7.515 | 7.515 | 7.520 | 3/15/00 | 6.723 | 6.723 | 6.723 | 6.723 |
| 12/16/97 | 8.005 | 8.005 | 8.004 | 8.002 | 6/15/00 | 6.710 | 6.710 | 6.710 | 6.710 |
| 10/7/98 | 8.233 | 8.255 | 8.247 | 8.229 | 9/21/00 | 6.750 | 6.748 | 6.749 | 6.748 |
| 10/7/99 | 8.261 | 8.262 | 8.261 | 8.268 | 12/20/00 | 6.780 | 6.780 | 6.780 | 6.780 |
| 10/9/01 | 8.378 | 8.383 | 8.380 | 8.379 | 3/20/01 | 6.837 | 6.837 | 6.837 | 6.836 |
| 10/7/04 | 8.620 | 8.630 | 8.627 | 8.629 | 6/21/01 | 6.822 | 6.822 | 6.822 | 6.822 |
| 10/10/06 | 8.917 | 8.921 | 8.918 | 8.909 | 9/20/01 | 6.857 | 6.857 | 6.857 | 6.857 |
| 10/7/09 | 8.783 | 8.788 | 8.783 | 8.789 | 7/2/02 | 6.877 | 6.882 | 6.880 | 6.883 |
| 10/7/14 | 8.924 | 8.935 | 8.921 | 8.930 | 7/2/04 | 6.983 | 6.985 | 6.984 | 6.983 |
| 10/7/24 | 8.541 | 8.545 | 8.541 | 8.541 | 7/2/07 | 7.127 | 7.132 | 7.131 | 7.132 |
| | | | 1.705 | 0.740 | 7/2/09 | 7.318 | 7.320 | 7.319 | 7.318 |
| | | | 2.065 | 0.740 | 7/2/12 | 7.371 | 7.374 | 7.372 | 7.372 |
| | | | 1.083 | 0.739 | 7/3/17 | 7.428 | 7.434 | 7.427 | 7.432 |
| | | | 0.980 | 0.738 | 7/5/22 | 7.325 | 7.328 | 7.316 | 7.307 |
| | | | 1.250 | 0.723 | 7/2/27 | 6.808 | 6.804 | 6.800 | 6.806 |
| | | | 1.048 | 0.682 | 7/2/32 | 6.732 | 6.729 | 6.735 | 6.728 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0.750 | 0.615 | 7/2/37 | 6.977 | 6.977 | 6.977 | 6.980 |
| | | | 0.402 | 0.540 | | | | 19.703 | 0.464 |
| | | | 0.321 | 0.474 | | | | 2.381 | 0.464 |
| | | | 0.257 | 0.417 | | | | 0.196 | 0.464 |
| | | | 0.093 | 0.371 | | | | 0.633 | 0.464 |
| | | | 0.011 | 0.342 | | | | 0.835 | 0.462 |
| | | | 0.089 | 0.334 | | | | 0.623 | 0.442 |
| | | | 1.002 | 0.342 | | | | 0.430 | 0.410 |
| | | | 0.940 | 0.308 | | | | 0.461 | 0.378 |
| | | | 0.179 | 0.211 | | | | 0.426 | 0.340 |
| | | | 0.055 | 0.116 | | | | 0.232 | 0.296 |
| | | | 0.091 | 0.090 | | | | 0.095 | 0.257 |
| | | | 0.107 | 0.114 | | | | 0.159 | 0.228 |
| | | | 0.031 | 0.026 | | | | 0.224 | 0.204 |
| | | | 0.000 | -0.015 | | | | 0.110 | 0.178 |
| | | | -0.021 | 0.047 | | | | 0.049 | 0.157 |
| | | | | | | | | 0.135 | 0.142 |
| | | | | | | | | 0.175 | 0.128 |
| | | | | | | | | 0.085 | 0.112 |
| | | | | | | | | 0.040 | 0.096 |
| | | | | | | | | 0.076 | 0.085 |
| | | | | | | | | 0.067 | 0.074 |
| | | | | | | | | 0.064 | 0.065 |
| | | | | | | | | 0.067 | 0.069 |
| | | | | | | | | 0.048 | 0.049 |
| | | | | | | | | 0.017 | 0.015 |
| | | | | | | | | -0.006 | 0.019 |
| | | | | | | | | -0.063 | -0.089 |
| | | | | | | | | -0.058 | -0.074 |
| | | | | | | | | 0.018 | 0.023 |

## APPENDIX - PROOF OF LINEAR FORWARDS

I state in the text that the assumption of piece-wise linear continuous zeros implies piece-wise linear (but discontinuous) forwards, specifically the continuously-compounded forward rate between breaks i and i+1 is given by:

(9a) $f(b_i+x) = f_{i+1} + 2*(f_{i+1} - y_i)*[x+(b_i - b_{i+1})/2]/b_{i+1}$

or

(9b) $f(b_i+x) = [y_i(b_{i+1}-2b_i) + y_{i+1}b_i + 2x(y_{i+1}-y_i)]/(b_{i+1}-b_i)$

where

$y_i$, $y_{i+1}$ = zero rate at breaks i, i+1
$b_i$, $b_{i+1}$ = time to breaks i, i+1
x = time from break i; i.e. $0 \le x \le b_{i+1}-b_i$
$f_i$ = average forward rate (forward rate at the middle of the period)

Equation (9b) is derived as follows. The general relation between forward and zero rates is given in equation (2b) above:

(2b) $$y(t) = [\int_0^t f(u)\,du] / t$$

For the zero yields at the break points,

(10a) $$y_i = y(b_i) = \int_0^{bi} f(u)du / b_i$$

(10b) $$y_{i+1} = y(b_{i+1}) = \int_0^{bi+1} f(u)du / b_{i+1}$$

Between the break points, i.e. for $0 \le x \le b_{i+1}-b_i$,

$$y(b_i+x) = \int_0^{bi+x} f(u)du / (b_i+x)$$

Using (10a), we can write this as

(11) $$y(b_i + x) = \left[ b_i y_i + \int_{b_i}^{b_i + x} f(u)du \right] / (b_i + x) \quad .$$

The restriction of linearity in zeros means that

(12) $$y(b_i + x) = y_i\,[1 - x/(b_{i+1}-b_i)] + y_{i+1}x/(b_{i+1}-b_i)\ .$$

Combining equations (11) and (12) gives

(13) $$\left[b_i y_i + \int_{b_i}^{b_i+x} f(u)du\right] / (b_i + x) = y_i[1 - x / (b_{i+1} - b_i)] + y_{i+1}x / (b_{i+1} - b_i)$$

After simplifying and taking the derivative of both sides with respect to x, we arrive at equation (9b) given above.

Equation (9a) is obtained by expressing f(u) as

(14) $$f(u) = f_{i+1} + c^*[u - (b_i + b_{i+1})/2\ ] \qquad \text{for } b_i \le u \le b_{i+1}$$

which gives

(15) $$\int_{bi}^{bi+x} f(u)du \;=\; f_{i+1}{}^*x + c^*x^*(b_i-b_{i+1})/2 + c^*x^2/2\ .$$

Using (10b) we get

$$y_{i+1} \;=\; \int_0^{bi+1} f(u)du / b_{i+1} \;=\; [b_i\, y_i + \int_{bi}^{bi+1} f(u)du] / b_{i+1}\ .$$

Substituting expression (15) we get

(16) $$y_{i+1} = [b_i y_i + f_{i+1}(b_{i+1} - b_i)] / b_{i+1}\ .$$

Finally, substituting (15) and (16) into (13) and simplifying, we find that expression (13) holds for all values of x in the relevant range only when

(17) $$c = 2[f_{i+1} - y_i] / b_{i+1}\ ,$$

which leads to (9a) when $x=u-b_i$ .

Figure 7 - Instantaneous Forward Rates For 5 October 1994 - Zero to Thirty Years

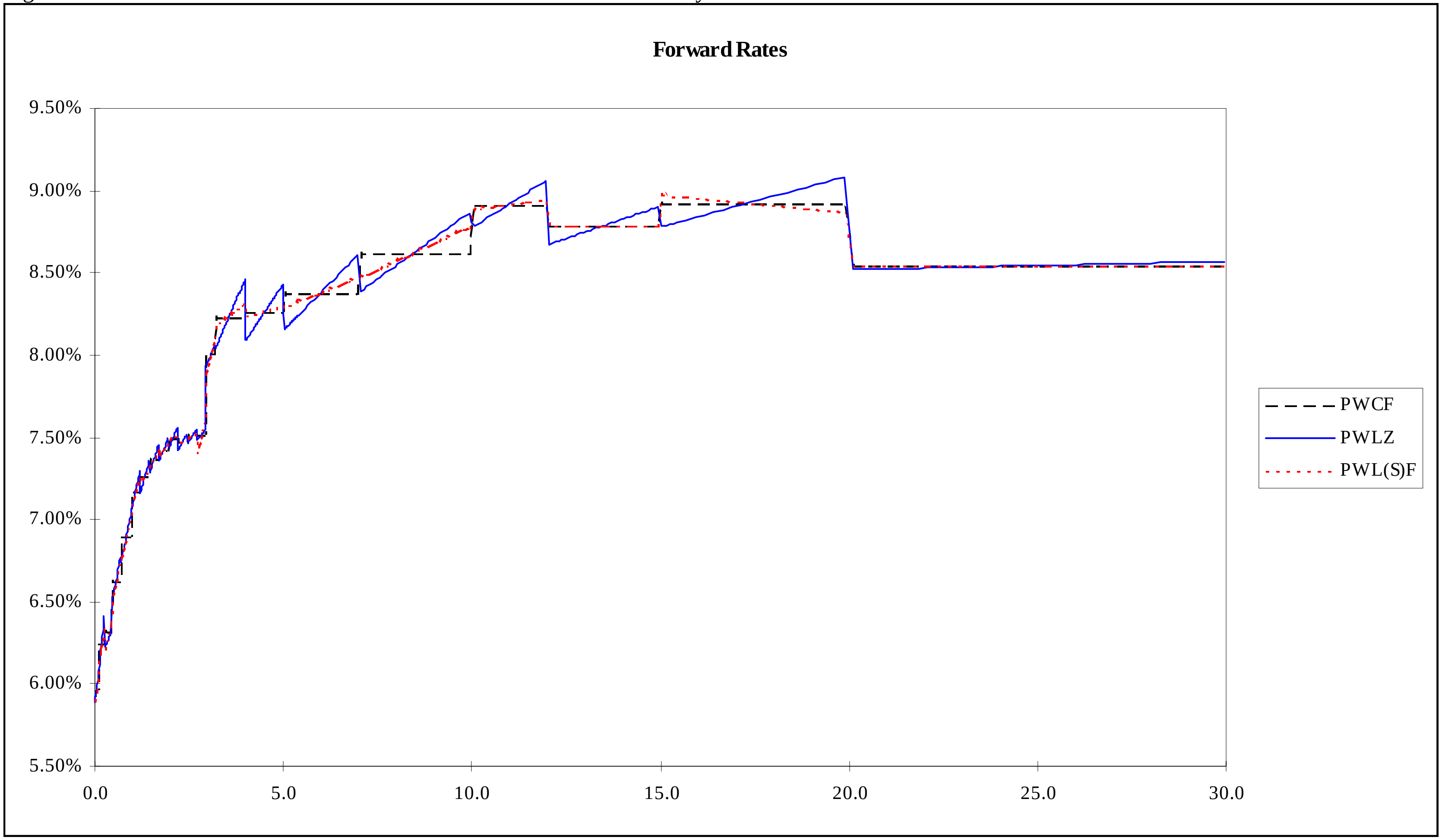


Figure 8 - Instantaneous Forward Rates For 5 October 1994 - Zero to Six Years

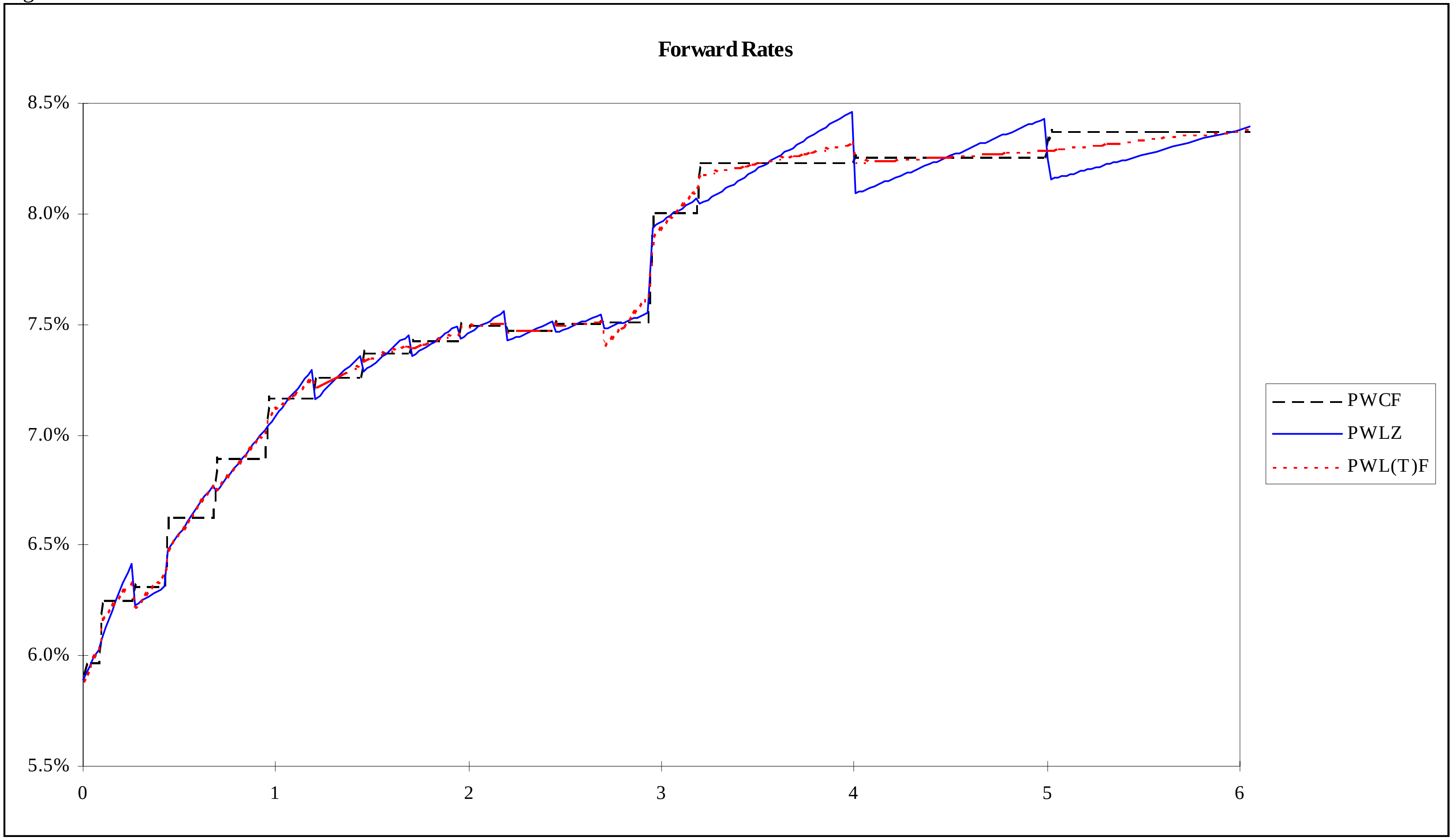


Figure 9 - Instantaneous Forward Rates For 30 June 1997 - Zero to Thirty Years

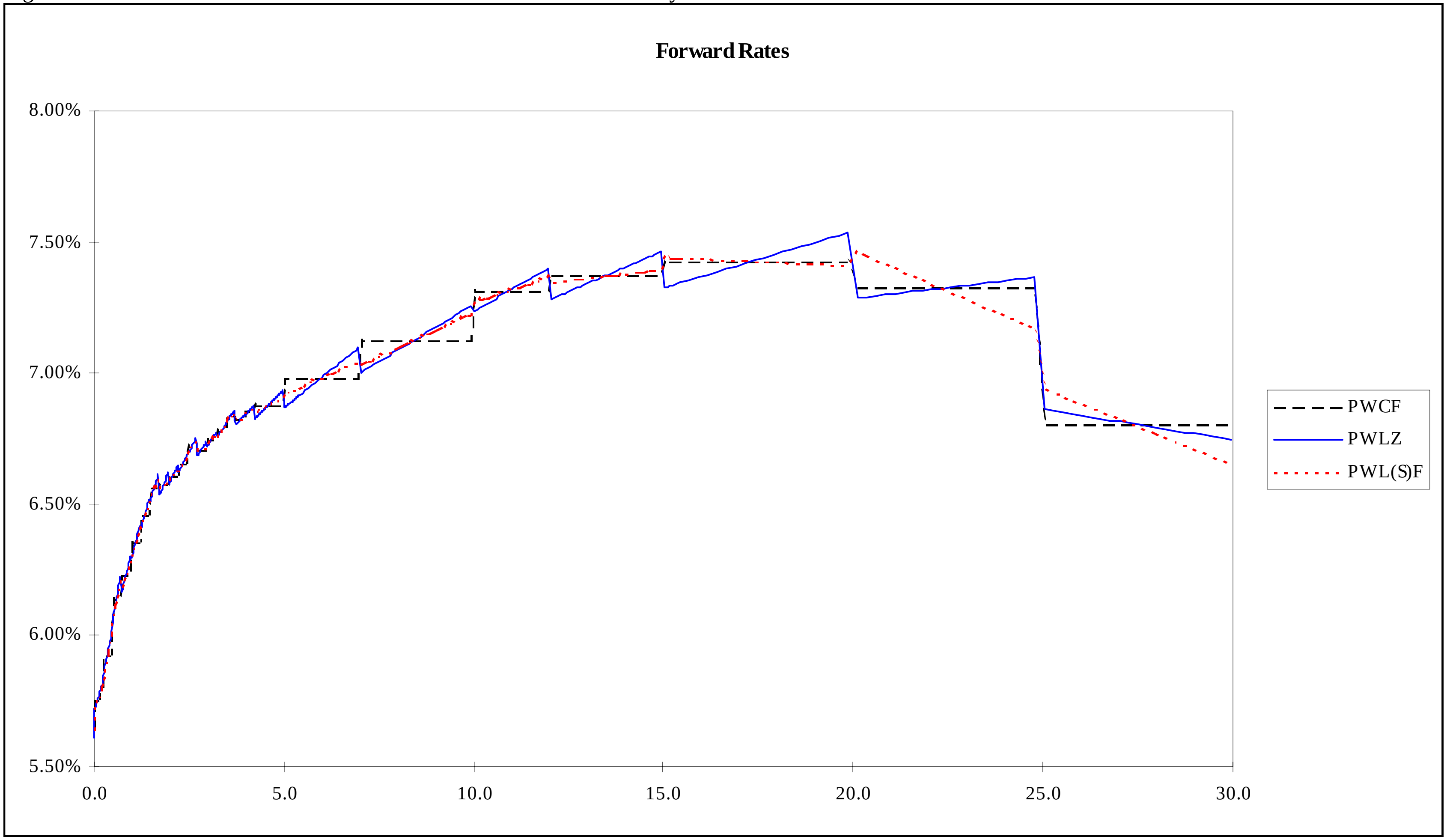


Figure 10 - Instantaneous Forward Rates For 30 June 1997 - Zero to Six Years

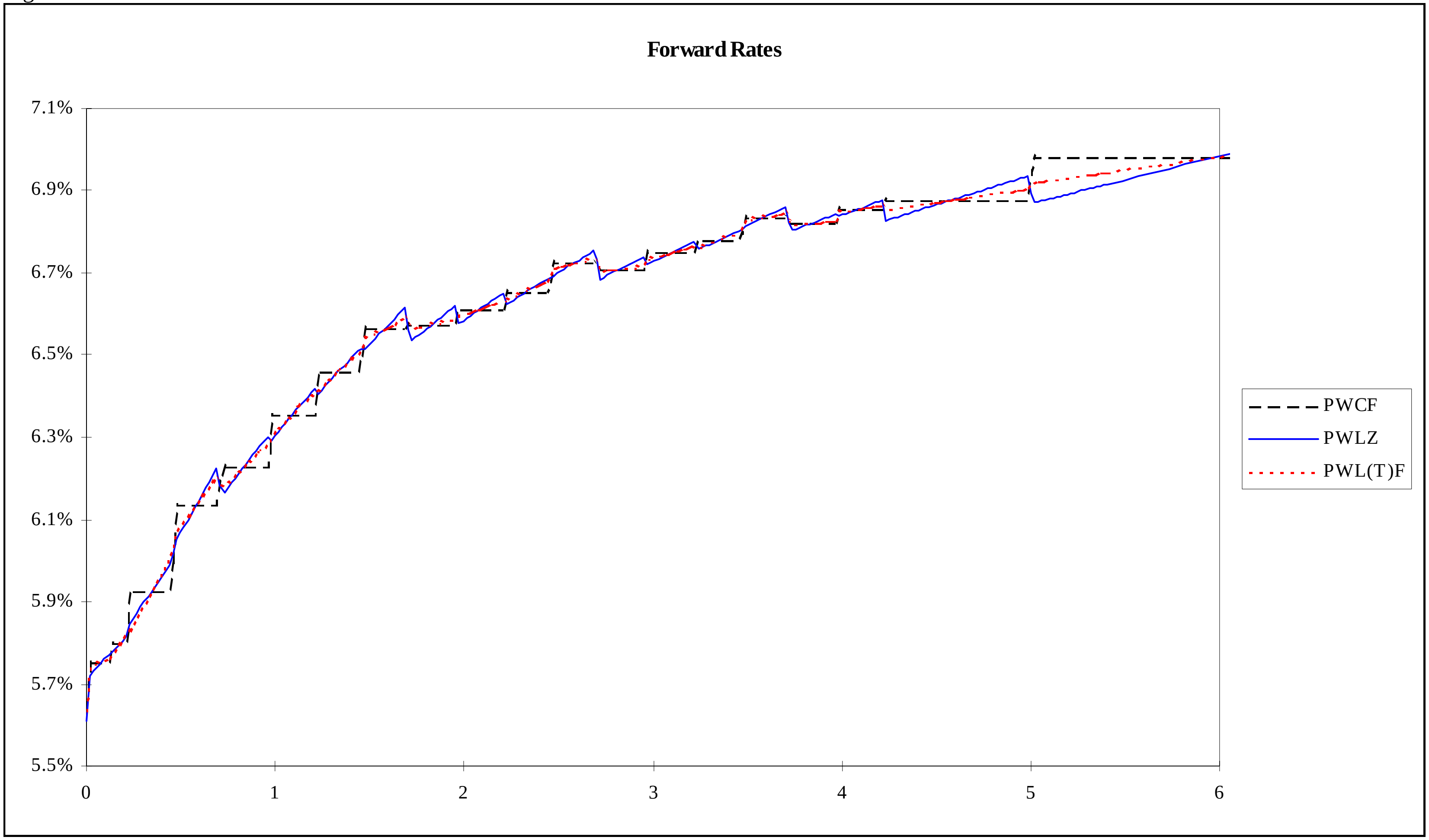


Figure 11 - Forward and Zero Rates for 5 October 1994

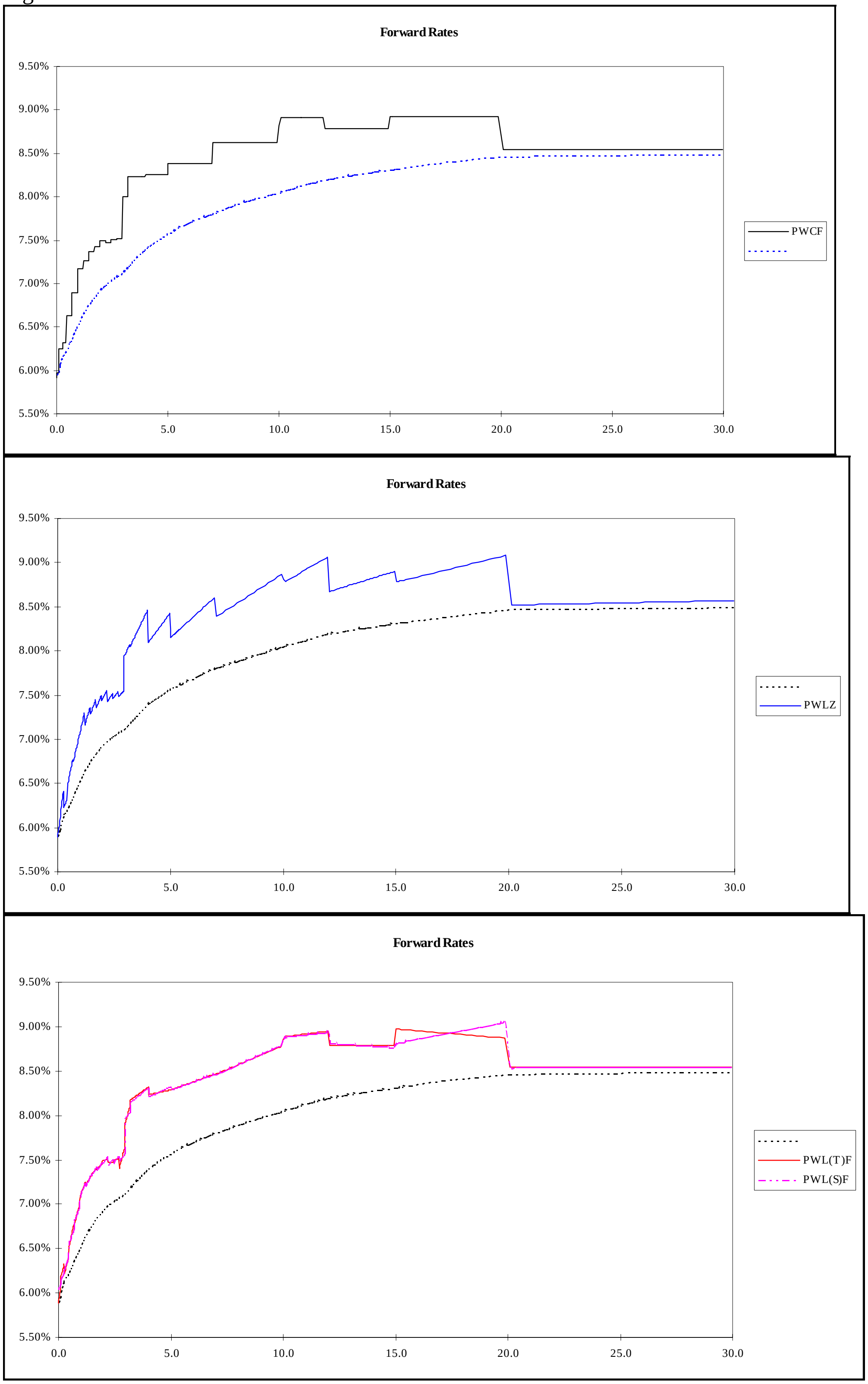


Figure 12 - Forward and Zero Rates for 30 June 1997

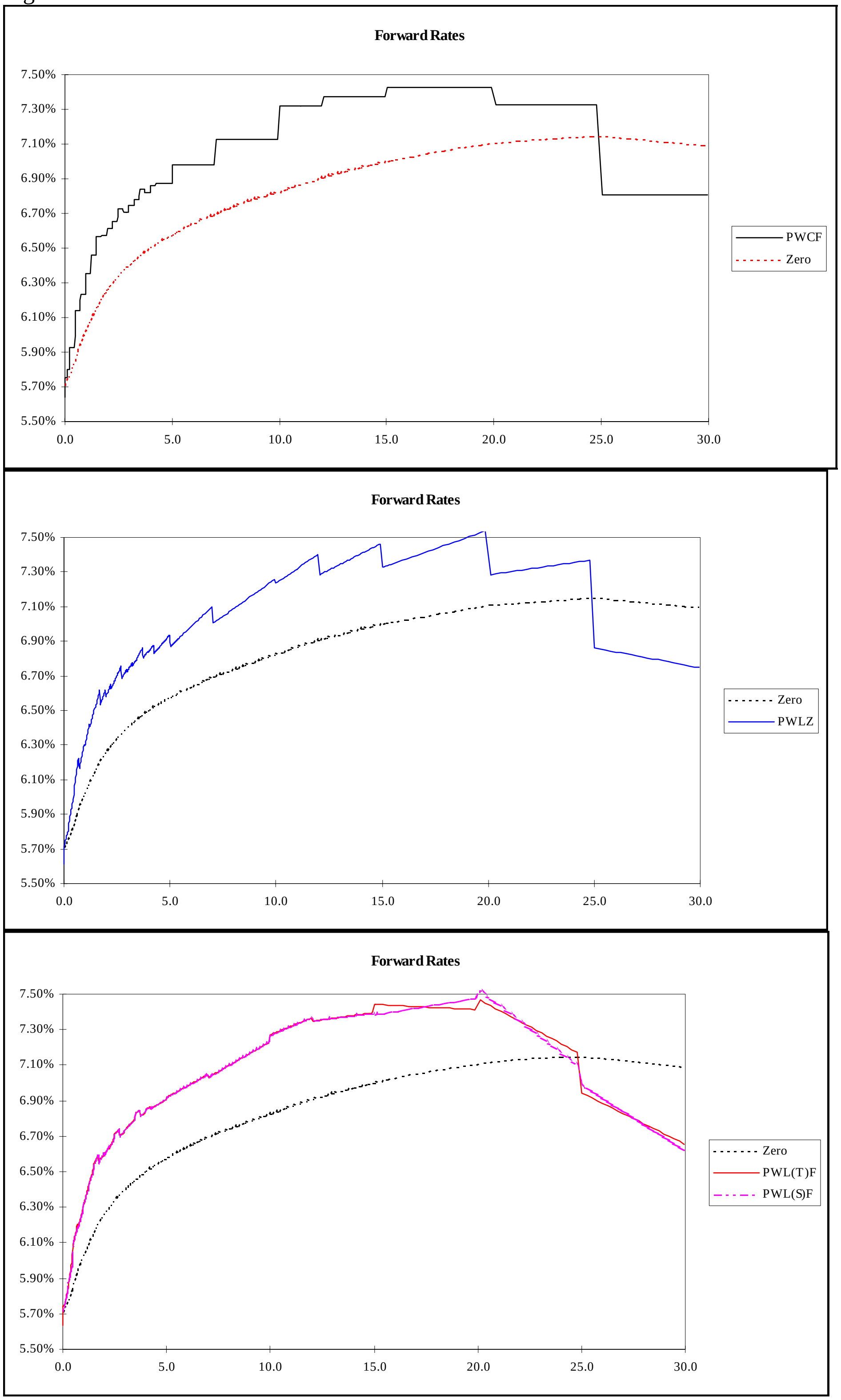


Figure 13 - Changes in Instantaneous Forward Rates Resulting from 1bp Shift in Seven Year Par Swap Input, June 1997

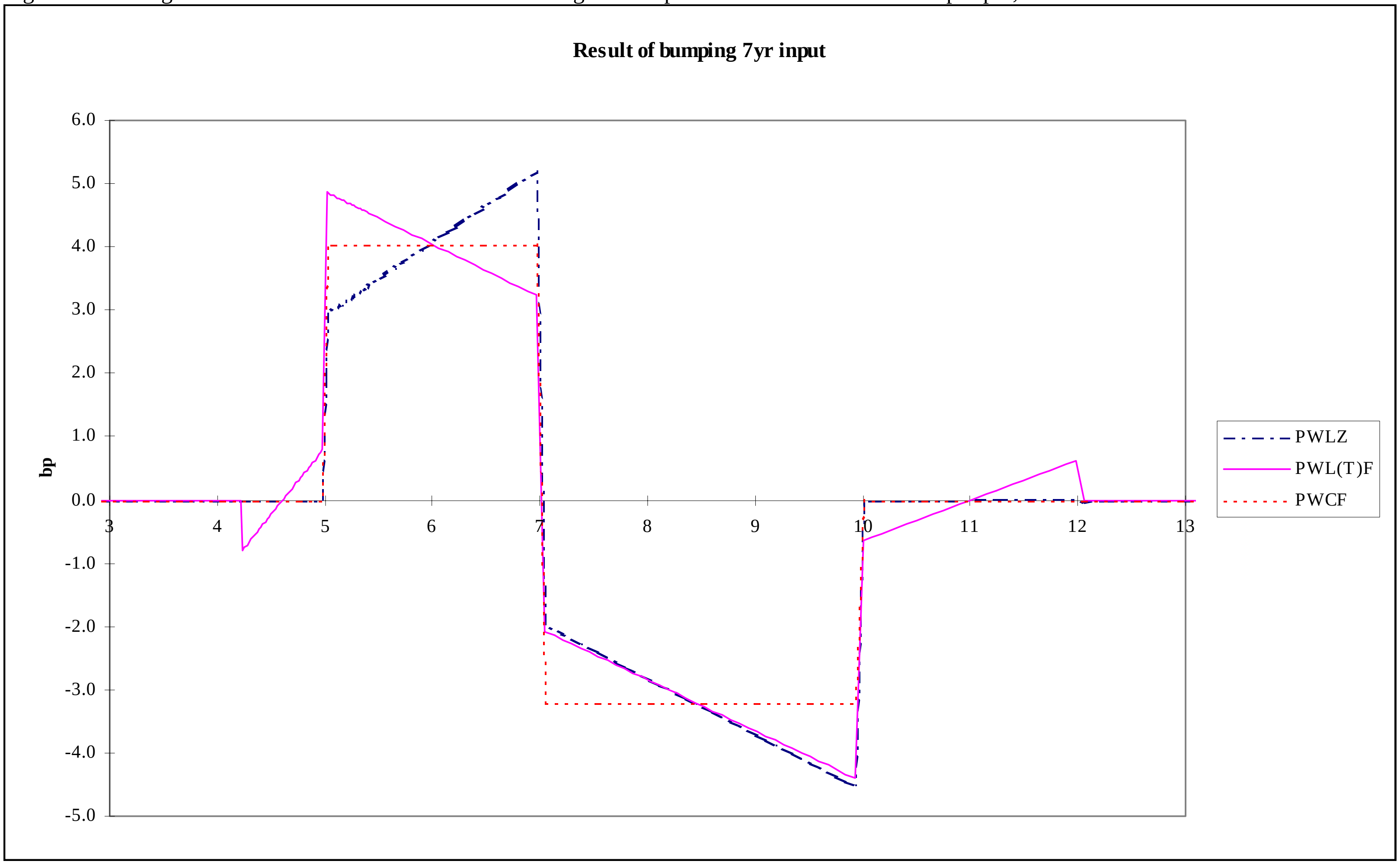
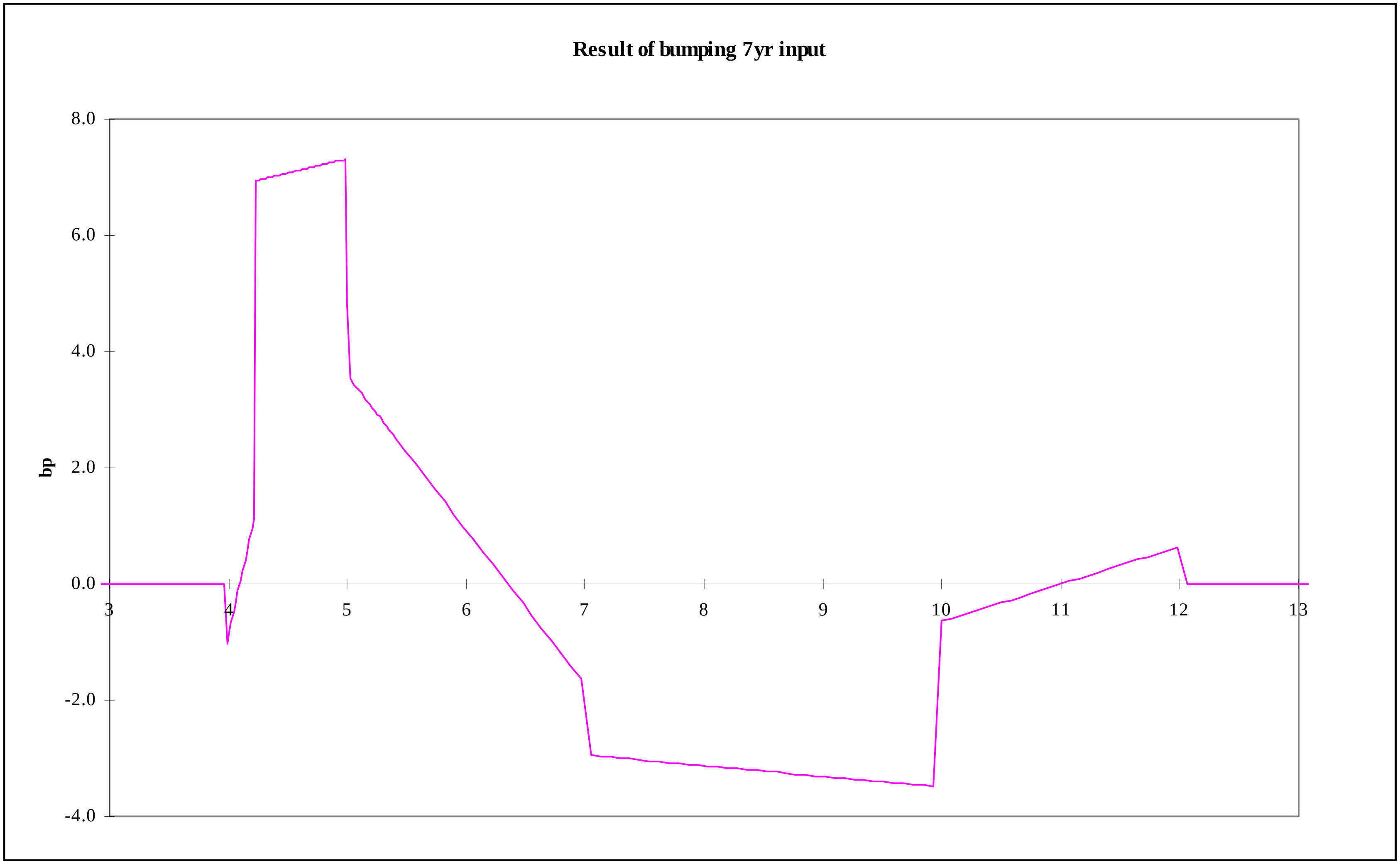



Figure 14 - Changes in Instantaneous Forward Rates Resulting from 1bp Shift in Five and Seven Year Par Swap Inputs, June 1997


Result of bumping 7yr input
8.0
6.0
4.0
2.0
0.0
-2.0
-4.0
bp
3
4
5
6
7
8
9
10
11
12
13

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